

INTRODUCTORY COURSE ON ℓ -ADIC SHEAVES AND THEIR RAMIFICATION THEORY ON CURVES

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ABSTRACT. These are the notes accompanying 13 lectures given by the authors at the Clay Mathematics Institute Summer School 2014 in Madrid. They give an introduction into the theory of ℓ -adic sheaves with emphasis on their ramification theory on curves.

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1. INTRODUCTION

These are the notes accompanying 13 lectures given by the authors at the Clay Mathematics Institute Summer School 2014 in Madrid. The goal of this lecture series is to introduce the audience to the theory of ℓ -adic sheaves with emphasis on their ramification theory. Ideally, the lectures and these notes will equip the audience with the necessary background knowledge to read current literature on the subject, particularly [16], which is the focus of a second series of lectures at the same summer school. We do not attempt to give a panoramic exposition of recent research in the subject.

Before giving an outline of this document, the authors wish to stress that there is *no original mathematical content in these notes, and that inaccuracies, omissions and errors are solely their responsibility*.

The text can roughly be divided into two parts: Sections 2 to 4 deal with the local theory and only assume a basic knowledge of commutative algebra, while the following sections are more global in nature and require some familiarity with algebraic geometry.

To introduce ideas, let p be a prime number and U a smooth, connected curve over the finite field \mathbb{F}_p . If $\ell \neq p$ is a second prime number, an ℓ -adic sheaf can be understood as a continuous representation $\rho: \pi_1^{\text{ét}}(U, u) \rightarrow \text{GL}_r(E)$, where E is a finite extension of the field of ℓ -adic numbers \mathbb{Q}_ℓ , and where $\pi_1^{\text{ét}}(U, u)$ is the étale fundamental group of U with respect to the base point u . This group is an algebraic variant of the usual fundamental group of a topological space; additionally, it carries a topology. We summarize its construction and properties in Section 6.

A representation ρ as above has a geometric interpretation (hence the word “sheaf”), which is explained in Section 7. If X is the unique smooth projective curve over \mathbb{F}_p containing U as an open dense subvariety, then one might ask whether ρ extends to a representation of $\pi^{\text{ét}}(X, u)$, i.e. whether ρ factors through the canonical map $\pi_1^{\text{ét}}(U, u) \rightarrow \pi_1^{\text{ét}}(X, u)$. If it does, ρ is called unramified. If it does not, then it is natural to ask whether it can be measured “how bad” the ramification of ρ is. The two important definitions in this context are *tame ramification* and *wild ramification*. Tame ramified ℓ -adic sheaves are much better understood than wildly ramified ones and they behave similarly to regular singular local systems on Riemann surfaces.

One of the main results presented in these notes is the construction and analysis of an invariant of ρ , locally at the finitely many closed points of $X \setminus U$, which measures how wild the ramification of ρ is; this is the so called Swan conductor. In the course of the construction, we give a proof of the Hasse-Arf theorem and we show that the Swan conductor arises from the character of a projective \mathbb{Z}_ℓ -representation, the Swan representation. See Sections 3 and 4.

The second main result gives a global, cohomological interpretation of the Swan conductor: This is the formula of Grothendieck-Ogg-Shafarevich, see Section 9.

The final sections survey generalizations of these notions to higher dimensional varieties. One approach based on ideas of Wiesend, further developed by Kerz, Schmidt, Drinfeld and Deligne is presented in Section 10. The idea is to study an ℓ -adic sheaf via its “skeleton”: If U is an algebraic variety over \mathbb{F}_p , consider the family $\{\varphi_C : C \hookrightarrow U\}$ of all curves lying on U , and let $\varphi_C^N : C^N \rightarrow C \hookrightarrow U$ be the composition with the normalization of C . If ρ is an ℓ -adic sheaf on U , then, roughly, its skeleton is the family of ℓ -adic sheaves $(\varphi_C^N)^* \rho$ on C^N . Using the ramification theory on curves one obtains in this way a ramification theory for ℓ -adic sheaves in higher dimensions. Recently, Deligne proved that on a smooth connected scheme over a finite field, there are only finitely many irreducible lisse $\bar{\mathbb{Q}}_\ell$ -representations with bounded rank and ramification, at least up to twist with a character of the Weil group of the ground field, see Theorem 10.18. One of the aims of these notes is to give the background necessary to understand the statement of this theorem. Notice, however, that its proof lies far beyond the scope of the material presented here and we refer the reader to [16] for details.

In the final section we give a very brief outlook on the higher-dimensional generalizations of the Grothendieck-Ogg-Shafarevich formula due to Kato-Saito. There are further generalizations of this formula due to Abbes and Saito who also develop a ramification theory in higher dimensions. We can say nothing about this, but give some references for further reading at the end of Section 11.

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2. INFINITE GALOIS THEORY

In this section we first define the notion of a profinite group and then briefly summarize Galois theory for infinite algebraic extensions. We will give more details in the more general situation of Section 6.

2.1. Profinite groups. We begin by recalling the notion of a projective limit; we mainly follow [10, Ch. V].

Definition 2.1. (a) A *directed set* is a set I together with a partial ordering \leq , such that for every pair $i, j \in I$ there exists $n \in I$ such that $n \geq i$ and $n \geq j$.

- (b) If \mathcal{C} is a category, then a *projective system in \mathcal{C} indexed by a directed set I* consists of the following data: For every $i \in I$ an object X_i of \mathcal{C} and for every $j \in I$ with $i \leq j$, a morphism $\varphi_{ji} : X_j \rightarrow X_i$, such that if $i \leq j \leq n$, the diagram

$$\begin{array}{ccc} X_n & & \\ \downarrow \varphi_{nj} & \searrow \varphi_{ni} & \\ & & X_i \\ & \nearrow \varphi_{ji} & \\ X_j & & \end{array}$$

commutes.

- (c) If $\{(X_i)_{i \in I}, (\varphi_{ji})_{i \leq j \in I}\}$ is a projective system, let \mathcal{C}' be the following category: Its objects are tuples $(X, (q_i)_{i \in I})$, where X is an object of \mathcal{C} and $q_i : X \rightarrow X_i$ morphisms in \mathcal{C} , such that for every $i \leq j \in I$, the diagram

$$\begin{array}{ccc} & & X_j \\ & \nearrow q_j & \downarrow \varphi_{ji} \\ X & & \\ & \searrow q_i & \\ & & X_i \end{array}$$

commutes. A morphism $(X, (q_i)_{i \in I}) \rightarrow (X', (q'_i)_{i \in I})$ in \mathcal{C}' is a morphism $f : X \rightarrow X'$ in \mathcal{C} , such that $q_i = q'_i f$ for all $i \in I$.

If \mathcal{C}' has a final object, then it is unique up to unique isomorphism and this final object is called the *projective limit* of the projective system $\{(X_i)_{i \in I}, (\varphi_{ji})_{i \leq j \in I}\}$, and usually denoted $(\varprojlim_I X_i, (\text{pr}_i)_{i \in I})$. Often the morphisms pr_i are omitted from the notation.

Example 2.2. (a) The simplest example of a projective system is a constant projective system: If X is an object of \mathcal{C} , and I a directed set, consider the projective system given by $X_i = X$, $\varphi_{ji} = \text{id}_X$. Clearly, $\varprojlim_I X_i \cong X$, and $\text{pr}_i = \text{id}$.

- (b) If \mathcal{C} is the category of groups, let $I = \mathbb{N}$ with its usual order, p a prime and $G_n := \mathbb{Z}/p^n \mathbb{Z}$. If $m \geq n$, then the projection map $\varphi_{mn} : \mathbb{Z}/p^m \rightarrow \mathbb{Z}/p^n$ makes $\{G_n, \{\varphi_{mn}\}_{n \leq m \in \mathbb{N}}\}$ into a projective system, and the projective limit $\varprojlim_{\mathbb{N}} G_n$ of this system is an abelian group, the *p -adic integers \mathbb{Z}_p* .

Moreover, as the maps φ_{mn} are maps of rings, the group \mathbb{Z}_p also carries a ring structure, which makes it a projective limit in the category of commutative rings.

- (c) Let \mathcal{C} be the category of groups and equip $I = \mathbb{N}$ with the partial order defined by $(n \leq m) :\Leftrightarrow n|m$. If n, m are integers such that $n|m$, then again we have a projection morphism $\varphi_{mn} : \mathbb{Z}/m \twoheadrightarrow \mathbb{Z}/n$, and $\{(\mathbb{Z}/n)_{n \in \mathbb{N}}, (\varphi_{mn})_{n|m \in \mathbb{N}}\}$ is a projective system. Its projective limit is an abelian group denoted $\widehat{\mathbb{Z}}$. It is not difficult to check that $\widehat{\mathbb{Z}} \cong \prod_p \text{prime } \mathbb{Z}_p$.

As in the previous example, $\widehat{\mathbb{Z}}$ is also a projective limit in the category of commutative rings.

- (d) More generally: If \mathcal{C} is the category of groups, I a directed set and $\{(G_i)_{i \in I}, (\varphi_{ji})_{j \geq i \in I}\}$ a projective system in \mathcal{C} , then $\varprojlim_I G_i$ exists and it is a subgroup of the product $\prod_{i \in I} G_i$:

$$\varprojlim_I G_i = \left\{ (g_i) \in \prod_{i \in I} G_i \mid \forall i \leq j : \varphi_{ji}(g_j) = g_i \right\} \subseteq \prod_{i \in I} G_i. \quad (2.1)$$

The morphisms $\text{pr}_j : \varprojlim_I G_i \rightarrow G_j$ belonging to the datum of the projective limit are induced by the projection maps $\prod_{i \in I} G_i \rightarrow G_i$.

We are now interested in the category **TopGroup** of topological groups.

Proposition 2.3. *The projective limit of a projective system of topological groups exists. If the groups in the projective system are Hausdorff and quasi-compact, then so is the inverse limit of the system.*

Indeed, this follows from the description (2.1).

If G is a finite group, we consider it as a topological group by equipping it with the discrete topology.

Proposition 2.4. *Let I be a directed set and $\{(G_i)_{i \in I}, (\varphi_{ji})_{i \leq j \in I}\}$ a projective system of finite groups. Its projective limit $\varprojlim_I G_i$ computed in the category of abstract groups exists and it is a subgroup of the product $\prod_{i \in I} G_i$. If we equip each G_i with the discrete topology and $\varprojlim_I G_i$ with the topology induced by $\prod_{i \in I} G_i$, then $\varprojlim_I G_i$ is also a projective limit of the system $\{(G_i)_{i \in I}, (\varphi_{ji})_{i \leq j \in I}\}$ computed in the category of topological groups.*

Moreover, the topology on $\varprojlim_I G_i$ is the coarsest topology such that the projections pr_i are continuous.

In other words: We get the same result if we equip the abstract group $\varprojlim_I G_i$ with the topology induced by the product, or if we compute the projective limit in the category of topological groups, by considering the G_i as discrete groups.

Definition 2.5. A topological group G is called *profinite*, if it is isomorphic to the projective limit of a projective system of finite (discrete) groups in the category of topological groups.

Similarly, if p is a prime number, G is called *pro- p -group* if G is isomorphic to the projective limit of a projective system of finite p -groups.

Example 2.6. (a) The groups \mathbb{Z}_p and $\widehat{\mathbb{Z}}$ from Example 2.2 are profinite groups.

(b) The group \mathbb{Z}_p is a pro- p -group.

(c) If G is an abstract group, then we write

$$\widehat{G} := \varprojlim_{\substack{H \triangleleft G \\ (G:H) < \infty}} G/H,$$

where H runs through the directed set of normal subgroups of finite index of G . The profinite group \widehat{G} is called the *profinite completion* of G .

(d) Similarly, if p is a prime number, we write

$$\widehat{G}^{(p)} := \varprojlim_{\substack{H \triangleleft G \\ (G:H)=p\text{-power}}} G/H,$$

where H runs through the directed set of normal subgroups of finite p -power index. The pro- p -group $\widehat{G}^{(p)}$ is called the *pro- p -completion* of G .

In fact, there is a purely topological description of profinite groups:

Proposition 2.7 ([10, V, §1.4]). *A topological group is profinite if and only if it is Hausdorff, quasi-compact and totally disconnected.*

Exercise 2.8. If G is a topological group and U an open subgroup, show that U is also closed. Moreover, if G is compact, show that U has finite index in G .

Corollary 2.9. *If G is profinite, then*

$$G \cong \varprojlim G/U,$$

where U runs through the set of open normal subgroups of G .

Corollary 2.10. *If G is a profinite group and $H \subseteq G$ a closed normal subgroup, then H and G/H are both profinite.*

More precisely,

$$H \cong \varprojlim H/H \cap U \text{ and } G/H \cong \varprojlim G/UH,$$

where in both cases U runs through the set of open normal subgroups of G .

Remark 2.11. Some caution is required: If G is a profinite group, then it is not always true that the canonical map $G \rightarrow \widehat{G}$ of G into its profinite completion is an isomorphism. In other words, it is not true that every finite index subgroup is open. However, if G is a topologically finitely generated profinite group (i.e. if there exists a finitely generated dense subgroup of G), then every subgroup of finite index is open ([40]). For example, there exist infinitely many finite index subgroups of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which are not open. In fact this is true for the subfield of $\overline{\mathbb{Q}}$ spanned by $\sqrt{-1}$ and $\{\sqrt{p} \mid p \text{ prime}\}$ (exercise or [38, §7.]).

2.2. The Galois correspondence for infinite algebraic extensions.

In this section we recall the basic statements of Galois theory for infinite extensions, as it can be found, for example, in [39], [38] or [48]. For a more general treatment see Section 6.

We assume that the reader is familiar with Galois theory of finite extensions.

Let K be a field and L an algebraic extension of K . The extension L/K is called Galois if $L^{\text{Aut}_K(L)} = K$. In this case we also write $\text{Gal}(L/K) := \text{Aut}_K(L)$. Equivalently, L/K is Galois if and only if the minimal polynomial of every element $\alpha \in L$ splits into distinct linear factors over L . In particular, Galois extensions are separable.

Proposition 2.12. *If L/K is a Galois extension of fields, write*

$$\mathcal{G}_{L/K} := \{F \subseteq L \mid F/K \text{ is a finite Galois extension}\}.$$

Then $L = \bigcup_{F \in \mathcal{G}_{L/K}} F$, and the inclusion relation makes $\mathcal{G}_{L/K}$ into a directed set. If $F_1 \subseteq F_2 \in \mathcal{G}_{L/K}$, then restriction of automorphisms induces a homomorphism $\text{Gal}(F_2/K) \rightarrow \text{Gal}(F_1/K)$. This makes the set of groups $\text{Gal}(F/K)$, $F \in \mathcal{G}_{L/K}$, into a projective system, and we have

$$\text{Gal}(L/K) = \varprojlim_{F \in \mathcal{G}_{L/K}} \text{Gal}(F/K).$$

In particular, $\text{Gal}(L/K)$ is a profinite group.

Exercise 2.13. (a) Let p be a prime, $n \in \mathbb{N}$, $q = p^n$ and \mathbb{F}_q the field with q elements. Fix an algebraic closure $\overline{\mathbb{F}_q}$. Show that the profinite group $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is isomorphic to $\widehat{\mathbb{Z}}$.

(b) Let $\mathbb{Q}^{\text{cyc}} \subseteq \mathbb{C}$ be the smallest subfield containing all roots of unity. Is \mathbb{Q}^{cyc} a Galois extension of \mathbb{Q} ? If so, what is $\text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$?

Theorem 2.14 (Galois correspondence). *Let L/K be a Galois extension and $\text{Gal}(L/K)$ its Galois group. There is an order reversing bijection*

$$\begin{aligned} \{\text{subextensions of } L/K\} &\xrightarrow{\cong} \{\text{closed subgroups of } \text{Gal}(L/K)\} \\ F &\longmapsto \text{Gal}(L/F) \end{aligned}$$

Under this correspondence,

- (a) *finite subextensions of L/K correspond to open subgroups,*
- (b) *subextensions which are Galois over K correspond to closed normal subgroups.*

Remark 2.15. In Section 6 we will see a vast geometric generalization of this correspondence. If F/K is a finite separable extension, then the associated morphism $\text{Spec } F \rightarrow \text{Spec } K$ will be interpreted as a (nontrivial!) covering space. The Galois group of F/K will then act as the group of deck transformations.

3. RAMIFICATION GROUPS AND THE THEOREM OF HASSE-ARF

In this rather lengthy section we summarize the ramification theory of finite separable extensions of a complete discretely valued field. Our main references are [46] and [39]. If L/K is such an extension, assume that the associated extension of residue fields is also separable. In this case we will construct two descending filtrations on the Galois group $G := \text{Gal}(L/K)$: the *lower numbering filtration* $\{G_u\}_{u \in \mathbb{Z}_{\geq -1}}$ and the *upper numbering* $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$. The subgroups appearing in both filtrations are the same, but the lower numbering is adapted to taking subgroups, while the upper numbering is adapted to taking quotients of G , i.e. if $H \triangleleft G$ is a normal subgroup, then $H_u = G_u \cap H$, and $(G/H)^v = G^v / (H \cap G^v)$.

The numbers $\lambda \in \mathbb{R}_{\geq -1}$, for which $G^\lambda \neq G^{\lambda+\varepsilon}$ for all $\varepsilon > 0$, are called *jumps* or *breaks* of the filtration. The theorem of Hasse-Arf (Theorem 3.63) states that the jumps are integers if G is abelian. This theorem is a crucial

ingredient for the ramification theory of ℓ -adic representations developed in the following sections.

3.1. Discretely valued fields and discrete valuation rings. We first recall some basic material on discrete valuations.

Definition 3.1. Let K be a field. A *discrete valuation* is a surjective homomorphism of abelian groups

$$v : K^\times \rightarrow \mathbb{Z}$$

such that $v(x + y) \geq \min\{v(x), v(y)\}$, for all $x, y \in K^\times$. We extend v to K , with the convention that $v(0) = \infty$.

If K is equipped with a discrete valuation v , then (K, v) is called a *discretely valued field*.

Remark 3.2. Some authors do not require a valuation to be surjective, but only nontrivial.

Exercise 3.3. Let (K, v) be a discretely valued field and $x, y \in K$. Show that $v(x + y) = \min\{v(x), v(y)\}$ if $v(x) \neq v(y)$.

If (K, v) is a discretely valued field, then the subset $A := A_K := \{x \in K \mid v(x) \geq 0\}$ is a commutative ring with unique maximal ideal $\mathfrak{m} := \mathfrak{m}_K := \{x \in A \mid v(x) > 0\}$. The local ring A is called *valuation ring of K* , and it is in fact a *discrete valuation ring*, i.e. a local principal ideal domain, which is not a field. A generator π of \mathfrak{m}_K is called *uniformizer of A , or uniformizer of K* . With our definitions, we always have $v(\pi) = 1$. The *residue field* A/\mathfrak{m}_K is also said to be the residue field of K .

Example 3.4. The three main examples to keep in mind are as follows:

- (a) Let $K = \mathbb{Q}$ be the field of rational numbers and p a prime number. An integer $a \in \mathbb{Z}$ can be uniquely written as $a = a'p^r$, with $a' \in \mathbb{Z}$ prime to p , and $r \in \mathbb{Z}_{\geq 0}$. We define $v_p(a) := r$, and for $\frac{a}{b} \in \mathbb{Q}$, $v_p(\frac{a}{b}) := v_p(a) - v_p(b)$. This defines a discrete valuation on \mathbb{Q} , which is called the *p -adic valuation*. The valuation ring associated with this valuation is $\mathbb{Z}_{(p)}$: the ring of rational numbers with denominators prime to p , and clearly p is a uniformizer of (\mathbb{Q}, v_p) . The residue field of (\mathbb{Q}, v_p) is \mathbb{F}_p , the finite field with p elements.
- (b) Let k be a field and C a connected, affine, normal, algebraic curve over k . This means that C is of the form $C = \text{Spec } A$, with A a normal, finite type k -algebra of Krull dimension 1. Let $K := k(C)$ be its function field, i.e. the local ring of C at its generic point, which amounts to saying that K is the field of fractions of A . If $c \in C$ is a closed point, we can associate with it a discrete valuation v_c on K : c corresponds to a maximal ideal $\mathfrak{m}_c \subseteq A$, and if $f \in A \setminus \{0\}$, there exists an integer $r \in \mathbb{Z}_{\geq 0}$, such that $f \in \mathfrak{m}_c^r \setminus \mathfrak{m}_c^{r+1}$. We define $v_c(f) := r$, and for $\frac{f}{g} \in K$, $v_c(\frac{f}{g}) = v_c(f) - v_c(g)$. The discrete valuation ring associated with v_c is $A_{\mathfrak{m}_c} = \mathcal{O}_{C,c}$, and its residue field $k(c)$ is a finite extension of k .

If C is a normal, proper curve, the converse is also true: Every discrete valuation of K is of the shape v_c for some closed point $c \in C$.

- (c) More generally, if k is a field and X a connected, finite type, normal k -scheme, to every codimension 1 point η of X (i.e. to every point η such that the closure of $\{\eta\}$ has codimension 1), one can attach a discrete valuation v_η in the same way as in the previous example. However, if $\dim X > 1$, there are discrete valuations on the function field $K = k(X)$ which do not arise from codimension 1 points of X ¹.

The globalization of the notion of a discrete valuation ring is called *Dedekind domain*.

Definition 3.5. A *Dedekind domain* A is a noetherian integral domain, which is not a field, such that the localization at each maximal ideal is a discrete valuation ring. This is equivalent to A being normal, noetherian and 1-dimensional (see [46, Ch. I, Prop. 4]).

Of course \mathbb{Z} is a Dedekind domain, and so is the ring A from Example 3.4, (b). In fact, geometrically, one can think of Dedekind domains as rings of functions on a curve.

3.2. Completion. A discretely valued field (K, v) is equipped with a natural topology, given by the “ultrametric” norm $\|\cdot\|_v$ defined by $\|x\|_v = \exp(-v(x))$. In fact, one could also take $\|x\| := a^{v(x)}$ for a any real number in $(0, 1)$, but the induced topology on K is independent of the choice of a . We will mostly be interested in discretely valued fields (K, v) which are complete with respect to this topology. Such fields are, naturally, called *complete discretely valued fields*. If (K, v) is complete, then its valuation ring A is complete in the \mathfrak{m}_K -adic topology, i.e. $A \cong \varprojlim_n A/\mathfrak{m}_K^n$.

There is a standard process of completing a discretely valued field (K, v) : Let K_v denote the ring of Cauchy sequences in K modulo the maximal ideal of sequences converging to 0. The valuation v can be extended uniquely to K_v according to the exercise below, so that (K_v, v) becomes a complete discretely valued field.

Exercise 3.6. Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in K not converging to 0. Show that the sequence of integers $(v(a_n))_{n \in \mathbb{N}}$ becomes stationary.

Example 3.7. We extend Example 3.4:

- (a) If $K = \mathbb{Q}$ is given the p -adic valuation v_p , then its completion is the field of p -adic numbers \mathbb{Q}_p . Its valuation ring is the ring of p -adic integers \mathbb{Z}_p . Note that the additive group underlying \mathbb{Z}_p coincides with the pro- p -completion of the abelian group \mathbb{Z} .
- (b) If k is a field and C a normal, connected algebraic curve over k , and $c \in C(k)$ a rational point, then the completion (K_c, v_c) of $K = k(C)$ with respect to v_c is noncanonically isomorphic to the field $k((t))$ of Laurent series over k . Geometrically, K_c should be thought of as an infinitesimally small, punctured disc around c .

¹There are even discrete valuations which do not arise from codimension 1 points on a different model of $k(X)$, see [33, Ch. 8, Thm. 3.26].

3.3. Extensions of discretely valued fields. If (K, v) is a complete discretely valued field with valuation ring A , and L/K a finite extension of fields, then there exists a unique discrete valuation v_L on L and an integer $e > 0$, such that for all $x \in K$, $v_L(x) = ev(x)$, and L is complete with respect to the discrete valuation v_L . Moreover, the integral closure B of A in L is a complete discrete valuation ring and a free A -module of rank $[L : K]$ ([46, Ch. II, Prop. 3]).

Definition 3.8. Keep the notations from the previous paragraph.

- By abuse of notation we say that v_L *extends* v .
- The integer e is called the *ramification index* of L over K .
- The extension L/K induces a finite extension of their residue fields. Its degree is denoted by f and called the *residue degree* of L over K .
- We say that L/K is *totally ramified* if $f = 1$.
- If $e = 1$ and if the extension of residue fields is separable, then L/K is called *unramified*.

In the situation of the definition, if $n = [L : K]$, then $n = ef$.

We globalize: Let (K, v) be a discretely valued field (not necessarily complete) with valuation ring A and L/K a finite extension. Denote by B the integral closure of A in L , then according to the Krull-Akizuki theorem ([39, Ch. I, Thm. 12.8]) B is a Dedekind domain, integral over A , but not necessarily local. In other words: The discrete valuation v can be extended to L , but there are finitely many distinct valuations w_1, \dots, w_r which extend it.

Lemma 3.9. *The Dedekind domain B is a finitely generated A -algebra, and hence a finitely generated A -module, in each of the following situations:*

- (a) *A is a complete discrete valuation ring (see the discussion before Definition 3.8).*
- (b) *L/K is separable ([46, Ch. I, Prop. 8]).*
- (c) *A is the localization of a finitely generated algebra over a field².*

In our applications, B will always be a finitely generated, hence finite, A -algebra, so from now on we tacitly assume this is the case.

Proposition 3.10 ([46, Ch. II, Thm. 1]). *Let (K, v) be a discretely valued field with valuation ring A and L/K a finite extension of degree n . Assume that the integral closure B of A in L is a finite A -algebra. Write w_1, \dots, w_r for the valuations of L extending v .*

- (a) *The canonical map $L \otimes_K K_v \rightarrow \prod_{i=1}^r L_{w_i}$ is an isomorphism.*
- (b) *L_{w_i}/K_v is a finite extension.*

Thus, for each i we obtain numbers n_i , e_i , and f_i : the degree, ramification index and residue degree of the extension L_{w_i}/K_v of complete discretely valued fields. As before we have $n_i = e_i f_i$ and

$$n = \sum_{i=1}^r n_i = \sum_{i=1}^r e_i f_i. \quad (3.1)$$

²or more generally, if A is a *Japanese/ $N-2$ ring*, see [35, (31.H)].

There is also an ideal theoretic interpretation of these numbers. The Dedekind domain B is a finite extension of the discrete valuation ring A , and the discrete valuations w_1, \dots, w_r correspond to the finitely many nonzero prime ideals \mathfrak{P}_i of B , i.e. $\mathfrak{P}_i = \{b \in B \mid w_i(b) > 0\}$. Since B is a Dedekind domain, it has the pleasant property that any ideal I can be uniquely written as a product

$$I = \mathfrak{P}_1^{s_1} \cdot \dots \cdot \mathfrak{P}_r^{s_r}.$$

In particular, if \mathfrak{m} is the maximal ideal of A , then $\mathfrak{m}B$ is a proper ideal in B , as the extension is integral, and hence $\mathfrak{m}B$ can be written as such a product of prime ideals. Indeed,

$$\mathfrak{m}B = \mathfrak{P}_1^{e_1} \cdot \dots \cdot \mathfrak{P}_r^{e_r},$$

where the e_i are precisely the ramification indices from above. Moreover, since B is 1-dimensional, the (nonzero) prime ideals \mathfrak{P}_i are maximal ideals, so we get finite extensions $k(v) = A/\mathfrak{m} \subseteq B/\mathfrak{P}_i$, and these extensions are precisely the degree f_i extensions of the residue fields of L_{w_i}/K_v .

Definition 3.11. Keep the notation from the previous paragraph.

- (a) If $r = 1$ and $f_1 = 1$, i.e. if there is only one valuation w_1 on L extending v and the residue extension is trivial, then L/K is called *totally ramified*.
- (b) If $e_i = 1$ and if the residue extension $A/\mathfrak{m}_K \subseteq B/\mathfrak{P}_i$ of L_{w_i}/K_{v_K} is separable, then L/K is called *unramified at \mathfrak{P}_i or at w_i* .
- (c) If L/K is unramified at w_1, \dots, w_r , then L/K is called *unramified*.

3.4. Discriminant and different. We start with a definition.

Definition 3.12. Let L/K be a finite extension of fields, or, more generally, let K be a ring and L a K -algebra which is free of finite rank as a K -module.

- For $x \in L$ define $N_{L/K}(x) \in K$ as the determinant of the K -linear endomorphism of L given by multiplication with x . The map $N_{L/K} : L^\times \rightarrow K^\times$ is called the *norm of L/K* .
- Similarly we define the *trace map of L/K* , $\text{Tr}_{L/K} : L \rightarrow K$, by defining $\text{Tr}_{L/K}(x)$ to be the trace of the K -linear endomorphism of L given by multiplication with x .
- For $x, y \in L$, we write $T_{L/K}(x, y) := \text{Tr}_{L/K}(xy)$.

An easy computation shows the following standard fact.

Proposition 3.13. Let L/K be a finite separable extension, and fix an algebraic closure \overline{K} of K . Let $\sigma_1, \dots, \sigma_r$ be the K -embeddings of L into \overline{K} . Then for $x \in L$, we have

$$N_{L/K}(x) = \prod_{i=1}^r \sigma_i(x)$$

and

$$\text{Tr}_{L/K}(x) = \sum_{i=1}^r \sigma_i(x).$$

Recall the following fact.

Proposition 3.14 ([7, IX, §2, Prop. 5]). *If R is a finite K -algebra, then R is a product of finite separable field extensions of K if and only if $\text{Tr}_{R/K}(-, -)$ is a nondegenerate bilinear form.*

Definition 3.15. Let A be a Dedekind domain with fraction field K and L a finite separable extension. Write B for the integral closure of A in L . Define the B -module

$$B^\vee := \{x \in L \mid \text{Tr}_{L/K}(xy) \in A \text{ for all } y \in B\}.$$

Proposition 3.16. *We have $B \subseteq B^\vee$, and B^\vee is a finitely generated B -submodule of L , i.e. a fractional ideal of B . The map*

$$B^\vee \rightarrow \text{Hom}_A(B, A), \quad x \mapsto (y \mapsto \text{Tr}_{L/K}(xy))$$

is an isomorphism of B -modules. The inverse of the fractional ideal B^\vee is an ideal in B .

Proof. Since L/K is separable, Proposition 3.14 shows that the map of K -vector spaces

$$L \rightarrow \text{Hom}_K(L, K), \quad x \mapsto (y \mapsto \text{Tr}_{L/K}(xy)) \quad (3.2)$$

is an isomorphism. On the other hand, every A -morphism $B \rightarrow A$ extends uniquely to a K -morphism $L \rightarrow K$, so $\text{Hom}_A(B, A)$ is a sub- A -module of $\text{Hom}_K(L, K)$, and it is precisely the image of B^\vee under (3.2). Since B is a finitely generated A -module, the same is true for $\text{Hom}_A(B, A)$ and hence for B^\vee .

Clearly, $B^\vee \supseteq B$, so $B = B^\vee \cdot (B^\vee)^{-1} \supseteq (B^\vee)^{-1}$. This finishes the proof. \square

Here is another characterization of B^\vee :

Lemma 3.17. *With the notations from above, B^\vee is the largest sub- B -module of L with the property that $\text{Tr}_{L/K}(B^\vee) \subseteq A$. In particular, if $\mathfrak{b} \subseteq L$ and $\mathfrak{a} \subseteq K$ are fractional ideals of B , A , then $\text{Tr}_{L/K}(\mathfrak{b}) \subseteq \mathfrak{a}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}B^\vee$.*

Proof. If E is a sub- B -module of L such that $\text{Tr}_{L/K}(E) \subseteq A$, then $E \subseteq B^\vee$, so the maximality statement is clear.

For the second statement, note that $\text{Tr}_{L/K}(\mathfrak{b}) \subseteq \mathfrak{a}$ if and only if $\text{Tr}_{L/K}(\mathfrak{a}^{-1}\mathfrak{b}) \subseteq A$, if and only if $\mathfrak{b} \subseteq \mathfrak{a}B^\vee$. \square

Definition 3.18. Keeping the notations from Definition 3.15, we write $\mathfrak{D}_{B/A} := (B^\vee)^{-1}$. This ideal of B is called the *different* of B/A .

The ideal $N_{L/K}(\mathfrak{D}_{B/A}) := \mathfrak{d}_{B/A}$ of A is called the *discriminant* of B/A .

Remark 3.19. Here the norm of an ideal is defined as follows: In a Dedekind domain, every ideal can be uniquely written as a product of prime ideals. If \mathfrak{P} is a nonzero prime ideal of B , then we define $N_{L/K}(\mathfrak{P}) := \prod_{\mathfrak{p}|\mathfrak{P}} \mathfrak{p}^{f_{\mathfrak{p}}}$, where $f_{\mathfrak{p}}$ is the degree of the residue extension $B/\mathfrak{P}/A/\mathfrak{p}$. Since we will only use these definitions in the case where L/K is an extension of *complete* discretely valued fields, we will only care about Dedekind rings which are discrete valuation rings, so we can forget the more complicated definition from above, and take the usual norm.

Proposition 3.20 (Transitivity of different and discriminant). *Let L/K and M/L be finite separable extensions, $A \subseteq K$ a Dedekind domain with $\text{Frac}(A) = K$, and B, C the integral closures of A in L , resp. M . Then*

$$\mathfrak{D}_{C/A} = \mathfrak{D}_{C/B} \mathfrak{D}_{B/A}$$

and

$$\mathfrak{d}_{C/A} = N_{L/K}(\mathfrak{d}_{C/B}) \mathfrak{d}_{B/A}^{[M:L]}.$$

Proof. The formula for the discriminant follows from the formula for the different by applying $N_{L/K}$. The formula for the different follows from the transitivity of the trace, i.e. from the fact that $\text{Tr}_{M/K} = \text{Tr}_{L/K} \circ \text{Tr}_{M/L}$ (exercise). \square

The different and discriminant indicate whether an extension is ramified or not. To see this, we now put ourselves in the complete local situation.

Lemma 3.21. *Let L/K be a finite separable extension of complete discretely valued fields, A the valuation ring of K , B its integral closure in L . Let x_1, \dots, x_r be a basis of B as an A -module. Let K^{sep} be a separable closure of K . Then*

$$\mathfrak{d}_{B/A} = \det((\text{Tr}_{L/K}(x_i x_j))_{ij}) = (\det((\sigma_i x_j)_{ij}))^2,$$

where σ_i runs through the finite number of embeddings of L in K^{sep} .

Proof. First, note that

$$\text{Tr}_{L/K}(x_i x_j) = \sum_k \sigma_k(x_i) \sigma_k(x_j) = (\sigma_1 x_i, \dots, \sigma_k x_i) \begin{pmatrix} \sigma_1 x_j \\ \vdots \\ \sigma_k x_j \end{pmatrix}.$$

It follows that

$$\det((\text{Tr}_{L/K}(x_i x_j))_{ij}) = (\det((\sigma_i x_j)_{ij}))^2.$$

The A -module $B^\vee = \text{Hom}_A(B, A)$ is spanned by the basis x'_1, \dots, x'_r , where $x'_i(x_j) = \text{Tr}_{L/K}(x_i x_j) = \delta_{ij}$. But B^\vee is a fractional ideal, so $B^\vee = (\beta)$, so $\beta x_1, \dots, \beta x_r$ is also an A -basis for B^\vee .

Write $d := \det((\text{Tr}_{L/K}(x_i x_j))_{ij})$, $d_\beta := \det((\text{Tr}(\beta x_i \beta x_j))_{ij})$. Then we compute

$$d_\beta = (\det((\sigma_i \beta x_j)_{ij}))^2 = N_{L/K}(\beta)^2 d = N_{L/K}(\mathfrak{D}_{B/A})^{-2} d.$$

If $d' := \det((\text{Tr}_{L/K}(x'_i x'_j))_{ij})$, then $d' d = 1$. Indeed, x_1, \dots, x_n and x'_1, \dots, x'_n are two bases of L over K . Let P be the associated base change matrix. We then have $P^t (\text{Tr}_{L/K}(x_i x_j))_{i,j} P = (\text{Tr}_{L/K}(x'_i x'_j))_{i,j}$, and $(\text{Tr}_{L/K}(x_i x_j))_{i,j} P = (\text{Tr}_{L/K}(x_i x'_j)) = \text{id}$, as $\text{Tr}_{L/K}(x_i x'_j) = \delta_{ij}$. Here $(-)^t$ denotes the transposed matrix. It follows that

$$d d' = \det((\text{Tr}_{L/K}(x_i x_j))_{i,j} \cdot P^t (\text{Tr}_{L/K}(x_i x_j))_{i,j} P) = \det(\text{Tr}_{L/K}(x_i x_j) P)^2 = 1.$$

Finally, we get the following equality of fractional ideals:

$$(d) = (d')^{-1} = (d_\beta)^{-1} = N_{L/K}(\mathfrak{D}_{B/A})^2 (d^{-1}).$$

Consequently $(d) = \mathfrak{d}_{B/A}$. \square

Proposition 3.22. *Let L/K be a finite separable extension of complete discretely valued fields. Then L/K is unramified if and only if $\mathfrak{D}_{B/A} = B$, if and only if $\mathfrak{D}_{B/A} = A$.*

Proof. Clearly, $\mathfrak{D}_{B/A} = B$ if and only if $\mathfrak{D}_{B/A} = A$.

By definition, the extension L/K is unramified if and only if $\mathfrak{m}_K B = \mathfrak{m}_L$, and if the residue extension is separable. Let x_1, \dots, x_r be a basis of B over A . Then the residue classes \bar{x}_i form a basis of $B/\mathfrak{m}_K B$ over A/\mathfrak{m}_K . Since $B/\mathfrak{m}_K B$ is a local ring, B is unramified over A , if and only if $B/\mathfrak{m}_K B$ is a separable A/\mathfrak{m}_K -algebra. By Proposition 3.14 this is equivalent to $\det(\mathrm{Tr}_{L/K}(\bar{x}_i \bar{x}_j)) \not\equiv 0 \pmod{\mathfrak{m}_K}$. According to the lemma this is equivalent to $\mathfrak{D}_{B/A} = A$. \square

We need two more facts about the different.

Theorem 3.23. *Let L/K be a finite separable extension of complete discretely valued fields. Assume that there exists $\alpha \in B$ which generates B as an A -algebra. If $f \in K[X]$ is the monic minimal polynomial of α , then $f \in A[X]$, and $\mathfrak{D}_{B/A} = (f'(\alpha))$.*

Proof. Let $f(X) \in K[X]$ be the monic minimal polynomial of α . The coefficients of $f(X)$ are symmetric functions in the roots of $f(X)$. Since α is integral over A , the same is true for all roots of $f(X)$ (in a sufficiently large extension of K), so $f(X) \in A[X]$. Let $b_0, \dots, b_{n-1} \in L$ such that

$$\frac{f(X)}{X - \alpha} := b_0 + b_1 X + \dots + b_{n-1} X^{n-1}. \quad (3.3)$$

The same argument as above shows that b_0, \dots, b_{n-1} are integral over B , hence elements of B .

Note that $1, \alpha, \dots, \alpha^{n-1}$ is a basis of B over A . We compute its dual basis of B^\vee .

Let $\alpha = \alpha_1, \dots, \alpha_n$ be the distinct roots of $f(X)$ (in a sufficiently large extension of K). Then $f'(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j)$. Hence, for $r \geq 0$ we compute

$$\left(\frac{f(X)}{X - \alpha_i} \right) (\alpha_j) \cdot \frac{\alpha_i^r}{f'(\alpha_i)} = \delta_{ij} \alpha_i^r.$$

Consequently, if $r = 0, \dots, n-1$, then

$$\mathrm{Tr}_{L[X]/K[X]} \left(\frac{f(X)}{X - \alpha} \frac{\alpha^r}{f'(\alpha)} \right) = \sum_{i=1}^n \frac{f(X)}{X - \alpha_i} \frac{\alpha_i^r}{f'(\alpha_i)} = X^r,$$

as the difference of both sides is a polynomial of degree $< n$ with zeroes $\alpha_1, \dots, \alpha_n$. In turn, we get

$$X^r = \mathrm{Tr}_{L[X]/K[X]} \left(\frac{f(X)}{X - \alpha} \frac{\alpha^r}{f'(\alpha)} \right) = \sum_{j=0}^{n-1} \mathrm{Tr}_{L/K} \left(\alpha^r \frac{b_j}{f'(\alpha)} \right) X^j,$$

which shows that $\frac{b_0}{f'(\alpha)}, \dots, \frac{b_{n-1}}{f'(\alpha)} \in B^\vee$ is the dual basis of $1, \alpha, \dots, \alpha^{n-1}$, and hence

$$B^\vee = \frac{1}{f'(\alpha)} \cdot (b_0 A + \dots + b_{n-1} A) \subseteq L.$$

Finally, from (3.3) we know that

$$\begin{aligned} b_{n-1} &= 1 \\ b_{n-i} - \alpha b_{n-i+1} &= a_{n-i+1} \text{ for } 1 < i < n \\ b_0 \alpha &= a_0. \end{aligned}$$

Inductively it follows that $\alpha^j \in \bigoplus_{i=0}^{n-1} b_i A$ for $j = 0, \dots, n-1$, so $B = \bigoplus_{i=0}^{n-1} b_i A$. It follows that

$$B^\vee = (f'(\alpha))^{-1},$$

which proves the claim. \square

The hypothesis of the theorem is always satisfied when the residue extension of L/K is separable.

Theorem 3.24 ([46, Ch. III, Prop. 12]). *Let L/K be a finite extension of complete discrete residue fields with separable residue extension. Then there exists an element $x \in B$, generating B as an A -algebra. Moreover, x can be chosen such that there exists a monic polynomial $R \in A[X]$ of degree f such that $R(x)$ is a uniformizer of B . If L/K is totally ramified, any uniformizer x generates B as an A -algebra.*

For future reference, we also prove:

Proposition 3.25. *Let L/K be a finite separable extension of complete discretely valued fields with separable residue extension. Write A for the valuation ring of K and $B = A[\alpha]$ for its integral closure in L .*

(a) *If the ramification index of L/K is e , then*

$$v_L(\mathfrak{D}_{B/A}) \geq e - 1$$

with equality if and only if L/K is tamely ramified (Definition 3.39).

(b) *If L/K is Galois with group G , then*

$$v_L(\mathfrak{D}_{B/A}) = \sum_{\sigma \in G \setminus \{1\}} i_G(\sigma)$$

where $i_G(\sigma) := v_L(\alpha - \sigma(\alpha))$, see Section 3.6.

Proof. After reducing to the totally ramified case, there exists a uniformizer $\alpha \in B$ such that $B = A[\alpha]$. Let f be the minimal polynomial of α . For both formulas we have to compute $v_L(f'(\alpha))$.

(a) We only sketch the proof, as we will not use this result in the sequel. We already saw that L/K is unramified if and only if $\mathfrak{D}_{B/A} = B$ if and only if $v_L(\mathfrak{D}_{B/A}) = 0$.

The minimal polynomial of α can be seen to be an Eisenstein polynomial $f(X) = X^e + a_{e-1}X^{e-1} + \dots + a_0$ with $a_i \in A$. We want to compute the valuation of

$$f'(\alpha) = e\alpha^{e-1} + (e-1)a_{e-1}\alpha^{e-2} + \dots + a_1.$$

The valuation of each summand is

$$v_i := v_L((e-i)a_{e-i}\alpha^{e-i-1}) = ev_K(e-i) + ev_K(a_{e-i}) + e-i-1$$

for $i = 0, \dots, e-1$. Note that the v_i are pairwise distinct, because they are modulo e . Hence

$$v_L(f'(\alpha)) = \min_{i=0, \dots, e-1} \{v_i\}$$

Note that for $i > 0$, $v_i = ev_K(e-i) + ev_K(a_{e-i}) + e-i-1 \geq e$, as f is an Eisenstein polynomial, and hence $v_K(a_{e-i}) > 0$. It follows that $v_L(f'(\alpha)) = e-1$ if and only if $v_0 = e-1$. But $v_0 = ev_K(e) + e-1$, so $v_0 = e-1$ if and only if $v_K(e) = 0$ if and only if L/K is tamely ramified.

If L/K is not tamely ramified, then $v_K(e) > 0$, so $v_0 = v_L(e) + e-1 \geq e$, so $e \leq v_L(f'(\alpha)) \leq v_L(e) + e-1$. This completes the proof.

- (b) This is much easier: $f'(\alpha) = \prod_{\sigma \in G \setminus \{\text{id}\}} (\alpha - \sigma\alpha)$; the formula for $v_L(f'(\alpha)) = v_L(\mathfrak{D}_{B/A})$ follows.

□

Remark 3.26. Theorem 3.23 allows us to characterize the different $\mathfrak{D}_{B/A}$ geometrically as follows: Let $\mu : B \otimes_A B \rightarrow B$ be the multiplication map $\mu(b_1 \otimes b_2) = b_1 b_2$ and I its kernel. Then I/I^2 is a $B \otimes_A B$ -module, so it also carries two B -module structures, obtained via the two maps $B \rightarrow B \otimes_A B$, $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$. It is not difficult to check that these two B -module structures on I/I^2 coincide. The B -module I/I^2 is denoted by $\Omega_{B/A}^1$, and it is called the module of *Kähler differentials*. The map $d : B \rightarrow \Omega_{B/A}^1$, $d(b) = b \otimes 1 - 1 \otimes b$ is an A -derivation.

If $B = A[X]/(f(X))$, write x for the image of X in B . Then $\Omega_{B/A}^1$ is generated by dx , and the annihilator of $\Omega_{B/A}^1$ is precisely the ideal $(f'(x))$. Indeed, there is an exact sequence of B -modules

$$(f(X))/(f(X))^2 \xrightarrow{d \otimes 1} \Omega_{A[X]/A}^1 \otimes_A B \rightarrow \Omega_{B/A}^1 \rightarrow 0,$$

and $df(X) = f'(X)dx$.

3.5. Galois theory of discretely valued fields. We are now going to study the Galois theory of finite extensions L/K , where K is equipped with a discrete valuation v_K . We start with the noncomplete setting.

Let (K, v_K) be a discretely valued field, L/K a finite separable extension. Let A be the valuation ring of K and B its integral closure in L . The separability of L/K implies that B is finitely generated over A (Lemma 3.9), so we can apply the results from Section 3.3.

Write w_1, \dots, w_r for the discrete valuations on L extending v_K . We saw in Proposition 3.10 that we have a canonical isomorphism of K_{v_K} -algebras $L \otimes_K K_{v_K} \xrightarrow{\cong} \prod_{i=1}^r L_{w_i}$. From this it follows that the extensions L_{w_i}/K_{v_K} are separable as well. Indeed, the extension L/K is defined by separable polynomials with coefficients in K , hence the finite K_{v_K} -algebra $L \otimes_K K_{v_K}$ is defined by separable polynomials with coefficients in K_{v_K} .

If L/K is in addition a Galois extension with group $G := \text{Gal}(L/K)$, then G acts on the set of valuations $\{w_1, \dots, w_r\}$ over v_K : If $\sigma \in G$, then $w_i \circ \sigma$ is a discrete valuation lying over v_K . Moreover this action is transitive ([46, Ch. I, Prop. 19], or [39, Ch. II, Prop. 9.1]), so the numbers e_i and f_i are

independent of $i \in \{1, \dots, r\}$; we just write e and f . The formula (3.1) becomes $n = efr$, where $n = [L : K]$.

Next, we define subgroups $D_{w_i} := \{\sigma \in G \mid w_i \circ \sigma = w_i\}$. By construction they are all conjugate subgroups of order ef of G ,

Definition 3.27. The subgroup D_{w_i} is called *decomposition group* of w_i .

In ideal theoretic language, if \mathfrak{P}_i is the prime ideal of B corresponding to w_i , then G acts transitively on the set $\{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$, and $D_{w_i} = \{\sigma \in G \mid \sigma(\mathfrak{P}_i) = \mathfrak{P}_i\} \subseteq G$.

Proposition 3.28. *If L/K is Galois, then L_{w_i}/K_{v_K} is Galois with group D_{w_i} .*

Proof. Indeed, we have seen that $[L_{w_i} : K_{v_K}] = ef = |D_{w_i}|$, and $D_{w_i} \subseteq \text{Aut}_{K_{v_K}}(L_{w_i})$. \square

Since we are mainly interested in the decomposition group, we now put ourselves in a complete local situation. Let L/K be a finite Galois extension of complete discretely valued fields. Write v for the valuation of K , v_L for the valuation of L , A for the valuation ring of K , and B for its integral closure in L . The Galois group $G = \text{Gal}(L/K)$ is equal to the decomposition group of v_L .

Next, consider the residue extension $k(v_L)/k(v)$. Since v_L is the unique valuation of L lying over v , we get a homomorphism of groups $G \rightarrow \text{Aut}_{k(v)}(k(v_L))$.

Proposition 3.29 ([46, Ch. I, Prop. 20]). *If the extension $k(v_L)/k(v)$ is separable, then it is Galois, and*

$$G \rightarrow \text{Gal}(k(v_L)/k(v))$$

is a surjective homomorphism of groups.

Its kernel is denoted by $I := I_L$.

Definition 3.30. The normal subgroup I of G is called *inertia group* of v_L .

The Galois correspondence shows that I corresponds to a field L^I lying between L and K ; the fixed field of I . It is Galois over K , and by construction this is the maximal subextension of L/K which is unramified. Indeed, $|G/I| = f = [L^I : K]$, so the ramification index of L^I/K is 1. On the other hand, the residue extension of L/L^I is trivial, so L/L^I is totally ramified.

The inclusion $I \subseteq G$ is the first step in the so called *ramification filtration*.

3.6. The ramification filtration — lower numbering. We continue with the previous setup: L/K is a finite Galois extension of complete discretely valued fields with separable residue extension. Let $G = \text{Gal}(L/K)$. We also write $p = \text{char}(k(v)) \geq 0$, A, B for the valuation rings of K, L , respectively, and $\mathfrak{m}_K, \mathfrak{m}_L$ for their maximal ideals.

Definition 3.31. Let $i \geq -1$ be an integer. Note that G acts on B/\mathfrak{m}_L^{i+1} . Define

$$G_i := \{\sigma \in G \mid \sigma \text{ acts trivially on } B/\mathfrak{m}_L^{i+1}\}.$$

The G_i form a descending chain of subgroups of G , and G_i is called the *i -th ramification subgroup* of G . This filtration is called the *ramification filtration* of G in the lower numbering.

We make the definition of the G_i a little bit more concrete:

$$G_i = \{\sigma \in G \mid \forall b \in B : v_L(\sigma(b) - b) \geq i + 1\}.$$

By Theorem 3.24, there exists $x \in B$ such that $B = A[x]$. Then we can also write

$$G_i = \{\sigma \in G \mid v_L(\sigma(x) - x) \geq i + 1\} \quad (3.4)$$

Indeed, x also generates B/\mathfrak{m}_L^{i+1} as an A -algebra, so σ acting trivially on B/\mathfrak{m}_L^{i+1} is equivalent to $v_L(\sigma(x) - x) \geq i + 1$.

Proposition 3.32 ([46, Ch. IV, Prop. 1]). *We have $G_{-1} = G$ and $G_0 = I$ the inertia group. The G_i are normal subgroups of G , and $G_i = \{1\}$ for $i \gg 0$*

Proof. The identifications of G_0 and G_{-1} follow directly from the definition, and so does the claim that G_i are normal. Finally, since G is a finite group, the description (3.4) shows that $G_i = \{1\}$ for i large enough. \square

The description (3.4) shows that if x is a generator of B as an A -algebra, then for $\sigma \in G \setminus \{\text{id}\}$, the integer

$$i_G(\sigma) := v_L(\sigma(x) - x) \quad (3.5)$$

is independent of the choice of x . We obtain a map $i_G : G \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$, which has the property that $G_i = i_G^{-1}([i + 1, \infty])$. As the G_i are normal subgroups, we also see that i_G is a class function, i.e. that it is constant on conjugacy classes.

Example 3.33. We begin to study an important example: Let k be an algebraically closed field of characteristic $p > 0$, and $k[[x]]$ the ring of formal power series in one indeterminate. Write $K := \text{Frac}(k[[x]]) = k((x))$, and consider the polynomial $t^p - t - \frac{1}{x} \in K[t]$. It is not difficult to see that it is irreducible. Indeed, if we make the change of coordinates $u = \frac{1}{t}$, we obtain an isomorphism of fields

$$L := K[u]/(u^p + xu^{p-1} - x) \cong K[t]/(t^p - t - \frac{1}{x}).$$

Note that $u^p + xu^{p-1} - x$ is an *Eisenstein polynomial* ([46, Ch. I, Prop. 17]). The extension L/K is a Galois extension with group \mathbb{F}_p , where $a \in \mathbb{F}_p$ acts via $t \mapsto t + a$, i.e. $u \mapsto \frac{u}{1+au}$. This extension is called *Artin-Schreier extension*.

Moreover, according to [46, Ch. I, Prop. 17], the integral closure of $k[[x]]$ in L is $K[u]/(u^p + xu^{p-1} - x)$, which is a discrete valuation ring with uniformizer $u = \frac{1}{t}$. The ramification index of this extension is p : $u^p = x(1 - u^{p-1})$, and $1 - u^{p-1}$ is a unit.

To determine the ramification groups of $\text{Gal}(L/K) = \mathbb{Z}/p\mathbb{Z}$, we compute for every $a \in \mathbb{F}_p \setminus \{0\}$:

$$v_L\left(\frac{u}{1+au} - u\right) = v_L\left(u\left(\frac{1 - (1+au)}{1+au}\right)\right) = 2.$$

It follows that $\mathbb{F}_p = G = G_0 = G_1 \not\supseteq G_2 = 0$.

Following [31], one similarly proves that for the $\mathbb{Z}/p^n\mathbb{Z}$ -Galois extension $L = K[t]/(t^{p^n} - t - x^{-m})$, one has $G_0 = G_1 = \dots = G_m \not\supseteq G_{m+1} = 0$, if $(m, p) = 1$.

Proposition 3.34 (Compatibility with taking subgroups). *Let H be a subgroup of G and L^H its fixed field. Then L/L^H is Galois with group H , and $H_i = G_i \cap H$ for all i .*

Proof. This is clear, since if x generates B over A , then x also generates B over the valuation ring of L^H . \square

Remark 3.35. (a) If we apply the proposition to the group G_0 , then L^{G_0} is the maximal unramified subextension of L/K , and L/L^{G_0} is totally ramified. To study the higher ramification groups G_i , $i > 0$, we may assume that L/K is totally ramified. In this case the degree of L/K is equal to the ramification index of L/K .
(b) If H is a normal subgroup of G , then G/H is the Galois group of the extension L^H/K . One might hope that the ramification filtration is compatible with quotients, i.e. that for $i \geq 0$, the image of G_i in G/H is the i -th ramification subgroup of G/H . This is not the case. The image of G_i in G/H is one of the ramification subgroups of G/H , but in general not $(G/H)_i$. The indexing has to be adjusted. To make this precise one introduces the *upper numbering filtration*, see Section 3.7.

Now let π be a uniformizer for L and assume that L/K is totally ramified. In this case $G = G_0$, and π generates B as A -algebra, see Theorem 3.24. For $\sigma \in G_0$, we see that $v_L(\sigma(\pi)) = 1$, so $\sigma(\pi)/\pi \in B^\times$. Moreover, by (3.4), G_i is characterized as the set of $\sigma \in G_0$, such that $\sigma(\pi) - \pi \in (\pi)^{i+1}$. Consequently,

$$G_i = \left\{ \sigma \in G \left| \frac{\sigma(\pi)}{\pi} \equiv 1 \pmod{\mathfrak{m}_L^i} \right. \right\}.$$

Definition 3.36. For $i > 0$, the set $U_L^i := 1 + \mathfrak{m}_L^i \subseteq B^\times$ is easily seen to be a subgroup, the group of i -th units. We define $U_L := U_L^0 := B^\times$.

This defines a descending filtration of the group of units U_L , and the observation that $\sigma(\pi)/\pi \in B^\times$ can be made more precise:

Proposition 3.37. *For $i \geq 0$, the assignment $\sigma \mapsto \sigma(\pi)/\pi$ induces an injective homomorphism of groups*

$$G_i/G_{i+1} \hookrightarrow U_L^i/U_L^{i+1},$$

which is independent of the choice of the uniformizer π .

Proof. We have seen that $\sigma(\pi)/\pi \in U_L^i$ if $\sigma \in G_i$. If τ is a second uniformizer, then $\tau = u\pi$ with $u \in B^\times$, and

$$\frac{\sigma(\tau)}{\tau} = \frac{\sigma(u)}{u} \cdot \frac{\sigma(\pi)}{\pi}.$$

If $\sigma \in G_i$, then by definition $\sigma(u) \equiv u \pmod{\mathfrak{m}_L^{i+1}}$, and hence $\sigma(u)/u \in U_L^{i+1}$, as $u \in B^\times$. We obtain a map $G_i \rightarrow U_L^i/U_L^{i+1}$, independent of the choice of π . This is a group homomorphism: If $\sigma, \sigma' \in G_i$, then

$$\frac{\sigma\sigma'(\pi)}{\pi} = \frac{\sigma'(\pi)}{\sigma'(\pi)} \cdot \frac{\sigma(\sigma'(\pi))}{\pi} = \frac{\sigma'(\pi)}{\pi} \cdot \frac{\sigma(\sigma'(\pi))}{\sigma'(\pi)},$$

and $\sigma'(\pi)$ also is a uniformizer of B .

Finally, $\sigma(\pi)/\pi \in U_L^{i+1}$ if and only if $\sigma \in G_{i+1}$, so the kernel of the map $G_i \rightarrow U_L^i/U_L^{i+1}$ is G_{i+1} . This completes the proof. \square

We can now use the fairly concrete structure of the groups U_L^i to get some information about the groups G_i :

Corollary 3.38. (a) G_0/G_1 is cyclic of order prime to $p = \text{char}(k(v_K))$.
 (b) If $p = 0$, then $G_i = 0$ for $i > 0$.
 (c) If $p > 0$, then for $i > 1$, the groups G_i are p -groups, and the quotients G_i/G_{i+1} are abelian p -groups.
 (d) G_0 is a semi-direct product of a cyclic group of order prime to p with a p -group. In particular, G_0 is solvable and G_1 is its unique p -Sylow subgroup.

Proof. (a) We know that the quotients G_i/G_{i+1} are isomorphic to subgroups of U_L^i/U_L^{i+1} . If $i = 0$, then $U_L^0/U_L^1 = k(v_L)^\times$. As every finite subgroup of the multiplicative group of a field is cyclic, we see that G_0/G_1 is cyclic. If $p > 0$, there are no elements of order p in $k(v_L)^\times$, so G_0/G_1 is cyclic of order prime to p .
 (b) Note that for $i > 0$, U_L^i/U_L^{i+1} is isomorphic to $\mathfrak{m}_L^i/\mathfrak{m}_L^{i+1}$, which is isomorphic to the additive group underlying $k(v_L)$. If $p = 0$, then the additive group $k(v_L)$ has no nontrivial elements of finite order, so $G_i/G_{i+1} = 0$ for all $i > 0$. As $G_i = 0$ for $i \gg 0$, it follows that $G_1 = 0$.
 (c) If $p > 0$ and $i > 0$, then again G_i/G_{i+1} is isomorphic to a finite subgroup of the additive group of the residue field $k(v_L)$, hence an abelian p -group. This implies that the order of G_1 is a p -power, so G_1 is a p -group.
 (d) We just have to show that the sequence

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_0/G_1 \longrightarrow 0$$

admits a splitting. But this follows from the fact that the orders of G_0/G_1 and G_1 are coprime. \square

The ramification filtration of the Galois group $G = \text{Gal}(L/K)$ allows us to “measure” the quality of ramification. If L/K is totally ramified, then its ramification index equals the degree $[L : K]$, the more G_i are nonzero, the “worse” is the ramification. We will make this more precise later on. For now, we content ourselves with the following definition:

Definition 3.39. A finite separable extension L/K of complete discretely valued fields is called *tamely ramified* if its ramification index is prime to the characteristic p of the residue field $k(v)$ of K , and if the extension of the residue fields is separable. Otherwise L/K is called *wildly ramified*.

These definitions are easily extended to the case of general separable extensions of discretely valued fields which are not necessarily complete: If (K, v) is a discretely valued field, L/K a finite separable extension and w a discrete valuation on L extending v (Definition 3.8), then $(L, w)/(K, v)$ is *tamely ramified* if the Galois extension L_w/K_v of the completions is tamely

ramified, i.e. if the ramification index of w over v is prime to p and if the residue extension is separable. Otherwise, $(L, w)/(K, v)$ is *wildly ramified*.

Exercise 3.40. Let L/K be a Galois extension of discretely valued fields, with separable residue extension and Galois group G . Show that L/K is tamely ramified if and only if $G_1 = 0$ if and only if $(|G_0|, p) = 1$.

3.7. The ramification filtration — upper numbering. We continue with the previous setup: Let L/K be a finite Galois extension of complete discretely valued fields with Galois group G and separable residue extension. Let $H \subseteq G$ be a normal subgroup, such that $G/H = \text{Gal}(L^H/K)$. As indicated in Remark 3.35, the image of the ramification filtration of G in G/H is the ramification filtration of $\text{Gal}(L^H/K)$, up to changing the numbering. We start this section by making this remark precise, before we then fix this numbering defect by introducing the *upper numbering for the ramification filtration*. Among other things, this will have the advantage that we can pass to the inverse limit to obtain a ramification filtration by closed subgroups of the absolute Galois group.

Recall that we defined a class function i_G on G , by setting $i_G(\sigma) = v_L(\sigma(x) - x)$ for $\sigma \neq \text{id}$, x a generator of B as A -algebra, and $i_G(\text{id}) = \infty$.

Proposition 3.41. *Let H be a normal subgroup of G , and e' the ramification index of L/L^H . We can then compute $i_{G/H}$ in terms of i_G :*

$$i_{G/H}(s) = \frac{1}{e'} \sum_{\sigma \mapsto s} i_G(\sigma).$$

Proof. Let A and B be the valuation rings of K, L and B' the valuation ring of L^H . By Theorem 3.24 we find $x \in B$, $y \in B'$ such that $B = A[x]$, $B' = A[y]$. By definition we have for $s \in G/H$:

$$e' i_{G/H}(s) = v_L(s(y) - y).$$

Let $\sigma \in G$ be a fixed element of the preimage of s . Then $\sigma|_{L^H} = s$ and every element of the preimage of s is of the form $\sigma\tau$ for $\tau \in H$. We show that the elements $\sigma y - y$ and $\prod_{\tau \in H} (\sigma\tau x - x)$ of B are associated, i.e. generate the same ideals.

Let $f(X) \in B'[X]$ be the minimal polynomial of x over L^H . Write $f(X) = X^n + b_{n-1}X^{n-1} + \dots + b_0$ with $b_i \in B'$. The coefficients of the polynomial $(\sigma f)(X) - f(X) \in B'[X]$ are $\sigma(b_i) - b_i$, $i = 0, \dots, n-1$. Let $B_i(Y) \in A[Y]$ be polynomials such that $B_i(y) = b_i$. Then $\sigma(b_i) - B_i(\sigma(y)) = 0$, so $Y - \sigma(y)$ divides the polynomial $\sigma(b_i) - B_i(Y) \in B'[Y]$, whence $y - \sigma(y)$ divides $\sigma(b_i) - b_i$. This shows that $\sigma y - y$ divides $(\sigma f)(x) - f(x) = \prod_{\tau \in H} (\sigma\tau x - x)$.

Conversely, let $G \in A[X]$ be a polynomial such that $G(x) = y$. Then x is a zero of $G(X) - y \in B'[X]$, so $G(X) - y = f(X)h(X)$ for some polynomial $h(X)$. Applying σ gives

$$G(X) - \sigma y = (\sigma f)(X)(\sigma h)(X)$$

and finally

$$y - \sigma y = G(x) - \sigma y = (\sigma h)(x) \prod_{\tau \in H} (\sigma\tau x - x).$$

□

For notational reasons, the following definition is convenient:

Definition 3.42. If $u \in [-1, \infty) \subseteq \mathbb{R}$, then $G_u := G_{\lceil u \rceil}$, where $\lceil u \rceil$ denotes the smallest integer $\geq u$.

We will use this notation in the proof of the following corollary.

Corollary 3.43 (Herbrand). *In the situation of the proposition, the ramification groups of G/H are the images of the ramification groups of G .*

Proof. Let $i \in \mathbb{Z}_{\geq -1}$, $\sigma_0 \in G_i$ and s_0 its image in G/H . The proposition allows us to compute $i_{G/H}(s_0)$:

$$i_{G/H}(s_0) = \frac{1}{e'} \sum_{\sigma \mapsto s_0} i_G(\sigma).$$

Write $H_j := G_j \cap H$ and $h_j := |H_j|$. We can rewrite the above formula:

$$i_{G/H}(s_0) = \frac{1}{e'} \sum_{j=-1}^{\infty} \sum_{\sigma \in H_j \setminus H_{j+1}} i_G(\sigma_0 \sigma).$$

Note that $i_G(\sigma_0 \sigma) \geq \min\{i_G(\sigma), i_G(\sigma_0)\}$ with equality if $i_G(\sigma) \neq i_G(\sigma_0)$, and that $i_G(\sigma) = j+1$ if $\sigma \in H_j \setminus H_{j+1}$. Using that $\sigma_0 \in G_i$, we compute

$$\begin{aligned} i_{G/H}(s_0) &= \frac{1}{e'} \left(\sum_{j=-1}^{i-1} (h_j - h_{j+1})(j+1) + \sum_{j \geq i} \sum_{\sigma \in H_j \setminus H_{j+1}} i_G(\sigma_0 \sigma) \right) \\ &\geq \frac{1}{e'} \left(\sum_{j=-1}^{i-1} (h_j - h_{j+1})(j+1) + (i+1) \cdot \sum_{j=i}^{\infty} (h_j - h_{j+1}) \right) \\ &= \frac{1}{e'} \sum_{j=0}^i h_j \end{aligned}$$

This last number is important and we give it a name:

$$\varphi_H(i) + 1 := \frac{1}{|G_0 \cap H|} \sum_{j=0}^i |G_j \cap H| \in \mathbb{Q}.$$

Our computation shows that the image $G_i H/H \subseteq G/H$ of G_i is a subgroup of $(G/H)_{\varphi_H(i)}$ (see Definition 3.42). Lets show that this is in fact an equality. Let $s \in (G/H)_{\varphi_H(i)}$, and let i' be maximal with the property that there exists a preimage σ_0 of s in $G_{i'}$. Let $\sigma \in H_j \setminus H_{j+1}$. If $j < i'$, then $i_G(\sigma_0 \sigma) = i_G(\sigma) = j+1$. If $j \geq i'$, then $i_G(\sigma_0 \sigma) \geq i' + 1$, and hence by the maximality of i' , $i_G(\sigma_0 \sigma) = i' + 1$. Our computation from above then shows

$$\begin{aligned} i_{G/H}(s) &= \frac{1}{e'} \left(\sum_{j=-1}^{i'-1} (h_j - h_{j+1})(j+1) + (i' + 1) \cdot \sum_{j \geq i'} (h_j - h_{j+1}) \right) \\ &= \frac{1}{e'} \sum_{j=0}^{i'} h_j \\ &= \varphi_H(i') + 1. \end{aligned}$$

We immediately see that $i_{G/H}(s) \geq \varphi_H(i) + 1$ implies that $i' \geq i$, so $s \in G_i H/H$. This completes the proof that $G_i H/H = (G/H)_{\varphi_H(i)}$. \square

To define the upper numbering filtration of G , we extend the definition of the function φ that appeared in the proof of Corollary 3.43.

Definition 3.44. As before, let L/K be a finite Galois extension of complete discretely valued fields with group G , and separable residue extension of degree f .

- (a) For $u \in [-1, 0)$, define $(G_0 : G_u) := (G_u : G_0)^{-1}$, i.e. $(G_0 : G_{-1}) = (G_{-1} : G_0)^{-1} = \frac{1}{f}$ and $(G_0 : G_u) = 1$ if $u \in (-1, 0)$.
- (b) Define the function $\varphi_{L/K} : [-1 : \infty) \rightarrow [-1, \infty)$ by

$$\varphi_{L/K}(u) = \int_0^u \frac{1}{(G_0 : G_t)} dt$$

If $u \geq 0$, then

$$\varphi_{L/K}(u) + 1 = \frac{1}{|G_0|} (|G_0| + |G_1| + \dots + |G_{[u]}| + (u - [u])|G_{[u]+1}|), \quad (3.6)$$

where $[u]$ is the largest integer $\leq u$. In particular, if $u = m \in \mathbb{Z}_{\geq 0}$, then

$$\varphi_{L/K}(m) + 1 = \frac{1}{|G_0|} \sum_{i=0}^m |G_i|.$$

This formula shows how the function $\varphi_{L/K}$ is related to the function φ_H from the proof of Corollary 3.43: $\varphi_{L/L^H} = \varphi_H$.

Using this notation, we restate the result of our calculation from the proof of Corollary 3.43 for future reference.

Corollary 3.45. *If $H \subseteq G$ is a normal subgroup, then for $u \in [-1, \infty) \subseteq \mathbb{R}$ we have*

$$G_u H / H = (G/H)_{\varphi_{L/L^H}(u)}.$$

The basic properties of $\varphi_{L/K}$ are easy to verify:

- Proposition 3.46.**
- (a) $\varphi_{L/K}$ is continuous, piecewise linear, increasing, and concave.
 - (b) $\varphi_{L/K}(0) = 0$.
 - (c) $\varphi_{L/K}$ is a homeomorphism of $[-1, \infty)$ onto itself.

Definition 3.47. The inverse of $\varphi_{L/K}$ is denoted $\psi_{L/K}$.

- Proposition 3.48.**
- (a) $\psi_{L/K}$ is continuous, piecewise linear, increasing, and convex.
 - (b) $\psi_{L/K}(0) = 0$.
 - (c) $\psi_{L/K}([-1, \infty) \cap \mathbb{Z}) \subseteq \mathbb{Z}$.

Exercise 3.49. Prove that $\psi_{L/K}(\mathbb{Z} \cap [-1, \infty)) \subseteq \mathbb{Z}$ (Use formula (3.6)).

Example 3.50. Lets compute $\varphi_{L/K}$ and $\psi_{L/K}$ in two simple special cases.

- (a) If L/K is unramified, then $\varphi_{L/K} = \psi_{L/K} = \text{id}$. Indeed, $(\varphi_{L/K})|_{[-1, 0)} = \text{id}|_{[-1, 0)}$ is always true. If $u \geq 0$, then

$$\varphi_{L/K}(u) = \frac{1}{|G_0|} (|G_1| + \dots + |G_{[u]}| + (u - [u])|G_{[u]+1}|) = u$$

as $|G_i| = 1$ for $i \geq 0$.

- (b) Now assume that L/K is Galois with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a prime number. Moreover, assume that $G = G_0 = G_1 = \dots = G_t$, and that $G_i = \{1\}$ for $i > t$. For $0 \leq u \leq t$, we compute

$$\varphi_{L/K}(u) = \frac{u \cdot \ell}{\ell} = u.$$

For $u > t$, we get

$$\varphi_{L/K}(u) = t + \frac{u - t}{\ell}$$

so

$$\psi_{L/K}(u) = \ell(u - t) + t,$$

for $u > t$.

This computation applies to Example 3.33.

The next proposition shows that the functions φ and ψ are transitive with respect to Galois subextensions.

Proposition 3.51. *With the notations from above, let $H \subseteq G$ be a normal subgroup. Then*

$$\varphi_{L/K} = \varphi_{L^H/K} \circ \varphi_{L/L^H} \text{ and } \psi_{L/K} = \psi_{L/L^H} \circ \psi_{L^H/K}.$$

Proof. Clearly it is enough to prove the claim about the function $\varphi_{L/K}$. Note that φ_{L/L^H} is piecewise linear and hence almost everywhere continuously differentiable. More precisely, it is differentiable on $[-1, \infty) \setminus \mathbb{Z}$, because if $u \notin \mathbb{Z}$, then the left and right derivatives of φ_{L/L^H} at u are equal to $(H_0 : H_u)^{-1}$. Similarly, ψ_{L/L^H} is continuously differentiable at all u , for which $\psi_{L/L^H}(u)$ is not an integer, and its derivative at such an u is

$$\psi'_{L/L^H}(u) = (\varphi'_{L/L^H}(\psi_{L/L^H}(u)))^{-1} = (H_0 : H_{\psi_{L/L^H}(u)}). \quad (3.7)$$

Next, recall Proposition 3.34, which tells us that $H_u = H \cap G_u$, for all $u \in [-1, \infty)$.

Using Corollary 3.45 we compute

$$\begin{aligned} \varphi_{L^H/K} \circ \varphi_{L/L^H}(u) &= \int_0^{\varphi_{L/L^H}(u)} \frac{dt}{((G/H)_0 : (G/H)_t)} \\ &= \int_0^{\varphi_{L/L^H}(u)} \frac{(H_0 : H_{\psi_{L/L^H}(t)})}{(G_0 : G_{\psi_{L/L^H}(t)})} dt. \end{aligned}$$

Now, with a certain degree of sloppiness, we use (3.7), to write

$$\int_0^{\varphi_{L/L^H}(u)} \frac{(H_0 : H_{\psi_{L/L^H}(t)})}{(G_0 : G_{\psi_{L/L^H}(t)})} dt = \int_0^{\varphi_{L/L^H}(u)} \frac{\psi'_{L/L^H}(t)}{(G_0 : G_{\psi_{L/L^H}(t)})} dt.$$

Of course ψ_{L/L^H} is not everywhere differentiable, so to be precise, one should do this calculation on every piece where it is differentiable and then sum up.

In the end, we obtain

$$\begin{aligned}\varphi_{L^H/K} \circ \varphi_{L/L^H} &= \int_0^{\varphi_{L/L^H}(u)} \frac{\psi'_{L/L^H}(t)}{(G_0 : G_{\psi_{L/L^H}(t)})} dt \\ &= \int_0^u \frac{dt}{(G_0 : G_t)} \\ &= \varphi_{L/K}(u)\end{aligned}$$

as claimed. \square

Now we have gathered all the facts to define the upper numbering for the ramification filtration.

Definition 3.52. We continue to use the notations from above. In particular $G = \text{Gal}(L/K)$ is the Galois group of a finite Galois extension of complete discretely valued fields with separable residue extension. For a real number $v \in [-1, \infty)$, define $G^v := G_{\psi_{L/K}(v)}$ or equivalently $G^{\varphi_{L/K}(u)} = G_u$.

Note that $G = G^{-1}$, $G^0 = G_0$.

As promised, the upper numbering filtration is compatible with passing to quotients:

Proposition 3.53 (Upper numbering filtration and quotients, Herbrand's Theorem). *Let $H \subseteq G$ be a normal subgroup. For every real number $v \in [-1, \infty)$, we have*

$$G^v H / H = (G/H)^v.$$

Proof. Most of the work has already been done in the proof of Corollary 3.43 and Proposition 3.51:

$$\begin{array}{ccc} G^v H / H & & (G/H)^{\varphi_{L^H/K}(\varphi_{L/L^H}(\psi_{L/K}(v)))} \xrightarrow{\text{Prop. 3.51}} (G/H)^v \\ \parallel \text{Def.} & & \parallel \text{Def.} \\ G_{\psi_{L/K}(v)} H / H & \xrightarrow{\text{Prop. 3.45}} & (G/H)_{\varphi_{L/L^H}(\psi_{L/K}(v))} \end{array}$$

\square

Proposition 3.53 allows us to define a ramification filtration of the absolute Galois group of K .

Definition 3.54. Let L/K be a (possibly infinite) Galois extension, such that the extension of the residue fields is separable. For a real number $u \in [-1, \infty)$, we define $\text{Gal}(L/K)^u := \varprojlim_{L'} \text{Gal}(L'/K)^u$, where L' runs through the set of finite Galois subextensions of L/K .

Remark 3.55. If the residue field of K is perfect this applies to a separable closure K^{sep} of K . We obtain a descending filtration of $\text{Gal}(K^{\text{sep}}/K)$.

3.8. The norm. Let L/K be an extension of discretely valued fields, and recall the definition of the norm map $N_{L/K}$ (Definition 3.12). Note that if L/K is Galois with Galois group G , then

$$N_{L/K} : L^\times \rightarrow K^\times, x \mapsto \prod_{\sigma \in G} \sigma(x).$$

Proposition 3.56. *The norm $N_{L/K}$ is a homomorphism of abelian groups. If L/K is a Galois extension of complete discretely valued fields with separable residue extension of degree f , then*

- (a) $v_K(N_{L/K}(x)) = f v_L(x)$
- (b) $N_{L/K}(B^\times) \subseteq A^\times$, where A, B are the valuation rings of K, L .

Proof. Exercise. □

Proposition 3.57 ([46, Ch. V]). *Let L/K be a totally ramified Galois extension of complete discretely valued fields and write $N := N_{L/K}$.*

- (a) *For $m \geq 0$, N induces a homomorphism of groups*

$$N : U_L^{\psi_{L/K}(m)} \rightarrow U_K^m$$

which is surjective if $G_{\psi_{L/K}(m)} = 0$ or if the residue field of K is algebraically closed.

- (b) *For $m \geq 0$, N induces a homomorphism of groups*

$$N : U_L^{\psi_{L/K}(m)+1} \rightarrow U_K^{m+1},$$

which is surjective if $G_{\psi_{L/K}(m)+1} = 0$ or if the residue field of K is algebraically closed.

- (c) *From the above we get homomorphisms*

$$N_m : U_L^{\psi_{L/K}(m)} / U_L^{\psi_{L/K}(m)+1} \rightarrow U_K^m / U_K^{m+1},$$

such that the sequence

$$0 \rightarrow G_{\psi_{L/K}(m)} / G_{\psi_{L/K}(m)+1} \xrightarrow{\theta_m} U_L^{\psi_{L/K}(m)} / U_L^{\psi_{L/K}(m)+1} \xrightarrow{N_m} U_K^m / U_K^{m+1}$$

is exact, where the map θ_m is the map defined in Proposition 3.37.

- (d) *N_m is surjective if one of the following holds:*

- *The residue field of K is algebraically closed.*
- *The residue field of K is perfect, and $G_{\psi_{L/K}(m)} = G_{\psi_{L/K}(m)+1}$.*
- *$G_{\psi_{L/K}(m)} = 0$.*

Proof. This is rather lengthy. We first prove a more explicit version of (d) by induction:

Claim. Let k be the residue field of L and K . We claim:

- Identifying U_L/U_L^1 and U_K/U_K^1 with k^\times , $N_0 : k^\times \rightarrow k^\times$ is given by the nonconstant monomial $X^{|G_0|}$.
- For $m \geq 1$, if we identify N_m with a map $k \rightarrow k$, then N_m is given by a nonconstant polynomial of the form $P_m(X)$ which is a linear combination of monomials of the form X^{p^h} , where $\text{char}(k) = p \geq 0$. Moreover $\deg P_m = |G_{\psi_{L/K}(m)}|$.
- Keeping the notations, if k is perfect, then there exists a polynomial Q_m with $P_m = (Q_m)^{p^h}$, where $p^h = |G_{\psi_{L/K}(m)+1}|$.

Part (d) follows from the claim. Indeed, if the residue field k is algebraically closed, this is obvious. If k is perfect and $m \geq 0$ such that $G_{\psi_{L/K}(m)} = G_{\psi_{L/K}(m)+1}$, we distinguish the cases $m = 0$ and $m > 0$. If $m = 0$, then $G_0 = G_1$ is a nontrivial p -group and the claim states that $N_0 : k^\times \rightarrow k^\times$

can be identified with the polynomial map $X^{|G_0|}$, which is surjective, as k is assumed to be perfect. If $m > 0$, then the claim shows that N_m can be identified with $P_m = (Q_m)^{|G_{\psi_{L/K}(m)+1}|}$ and that $\deg P_m = |G_{\psi_{L/K}(m)}|$ so $\deg Q_m = 1$. As k is assumed to be perfect, it follows that N_m is surjective. Finally, if $G_{\psi_{L/K}(m)} = 0$ then $m > 0$, as otherwise L/K would be trivial, and the claim implies that N_m can be identified with a nonconstant polynomial of degree 1, so N_m is surjective. This shows that the claim implies (d).

The surjectivity statements from (a) and (b) are implied by (d) using a completeness argument ([46, Ch. V, Lemma 2]).

Since the Galois group of L/K is solvable, it has a quotient which is cyclic of prime order. Using induction and the transitivity of the norm and ψ (Proposition 3.51), one reduces the proof of (c) and the claim to the case where the Galois group of L/K is $G = G_0 = \mathbb{Z}/\ell\mathbb{Z}$ (see [46, Ch. V, §6]). We assume that $G_0 = G_1 = \dots = G_t$, and $G_i = \{1\}$ for $i > t$.

In this case we already computed the functions $\varphi_{L/K}$ and $\psi_{L/K}$ in Example 3.50. We will use the following lemma:

Lemma 3.58. *Let B/A be the extension of discrete valuation rings attached to L/K . For brevity, we write $N_{L/K} := N$ and $\text{Tr} := \text{Tr}_{L/K}$.*

- (i) $\mathfrak{D}_{B/A} = \mathfrak{m}_L^{(t+1)(\ell-1)}$.
- (ii) If $n \geq 0$, then

$$\text{Tr}(\mathfrak{m}_L^n) = \mathfrak{m}_K^{\left\lfloor \frac{(t+1)(\ell-1)+n}{\ell} \right\rfloor}.$$

- (iii) If $x \in \mathfrak{m}_L^n$, then

$$N(1+x) \equiv 1 + \text{Tr}(x) + N(x) \pmod{\text{Tr}(\mathfrak{m}_L^{2n})}.$$

Proof. (i) Proposition 3.25, (b).

- (ii) Recall that if $\mathfrak{b} \subseteq L$ and $\mathfrak{a} \subseteq K$ are fractional ideals, then $\text{Tr}(\mathfrak{b}) \subseteq \mathfrak{a}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{D}_{B/A}^{-1}$ (Lemma 3.17). It follows that $\text{Tr}(\mathfrak{m}_L^n) \subseteq \mathfrak{m}_K^r = \mathfrak{m}_L^{\ell r}$ if and only if $r \leq \frac{(t+1)(\ell-1)+n}{\ell}$.

- (iii) Exercise (see [46, Ch. V, §3., Lemma 5]).

□

We can proceed with the proof:

$m = 0$: Clearly $N(U_L) \subseteq U_K$ and $N(U_L^1) \subseteq U_K^1$. The map $N_0 : k^\times \rightarrow k^\times$ can be identified with the polynomial X^ℓ . Let us determine the kernel of N_0 : Note that $\ell = p := \text{char}(k)$ if and only if $t > 0$, as G_1 is the unique p -Sylow subgroup of G_0 (Corollary 3.38). Thus, if $t = 0$, then the kernel of N_0 is of order ℓ and contains $\theta_0(G_0)$ which is also of order ℓ . If $t > 0$, then $\ell = p$ and hence N_0 is injective. Thus for $m = 0$ we have checked that (c) holds.

$1 \leq m < t$: In this case $\psi_{L/K}(m) = m$ (Example 3.50), so we first show that $N(U_L^m) \subseteq N(U_K^m)$. If $x \in \mathfrak{m}_L^m$, then $\text{Tr}(x) \in \mathfrak{m}_K^{m+1}$ according to Lemma 3.58, as

$$\left\lfloor \frac{(t+1)(\ell-1)+m}{\ell} \right\rfloor > \left\lfloor \frac{\ell m + (\ell-1)}{\ell} \right\rfloor \geq m.$$

This also shows that $\text{Tr}(\mathfrak{m}_L^{2m}) \subseteq \mathfrak{m}_K^{m+1}$, so Lemma 3.58 yields

$$N(1+x) \equiv 1 + N(x) \pmod{\mathfrak{m}_K^{m+1}}.$$

But $N(x) \in \mathfrak{m}_K^m$, so we see that $N(U_L^m) \subseteq U_K^m$, and also that $N(U_L^{m+1}) \subseteq U_K^{m+1}$.

To make N_m explicit, note that since $t > 0$ we have $\ell = p$, and if $x = u\pi^m$ with $u \in U_L$ and π a uniformizer, then $N(x) \equiv u^p N(\pi)^m \pmod{\mathfrak{m}_K^{m+1}}$ (Recall that $\sigma u - u \in \mathfrak{m}_L^{t+1}$ for all u). If τ is a uniformizer for K , then $N(\pi^m) = a_m \tau^m$, with $a_m \in U_K$. Then $N(1 + u\pi^m) \equiv 1 + a_m u^p \tau^m \pmod{\mathfrak{m}_K^{m+1}}$. Thus N_m can be identified with the polynomial $\alpha_m X^p$, where α_m is the image of a_m in k^\times . From this we also see that N_m is injective, and that it is surjective if k is perfect.

$m = t$: Let $x \in \mathfrak{m}_L^t$. Lemma 3.58 shows that

$$N(1+x) \equiv 1 + \text{Tr}(x) + N(x) \pmod{\mathfrak{m}_K^{t+1}}.$$

Moreover,

$$\left\lfloor \frac{(t+1)(\ell-1) + t}{\ell} \right\rfloor = t$$

and

$$\left\lfloor \frac{(t+1)(\ell-1) + t + 1}{\ell} \right\rfloor = t + 1$$

so $N(U_L^t) \subseteq U_K^t$ and $N(U_L^{t+1}) \subseteq U_K^{t+1}$. As $\psi_{L/K}(t) = t$, this is what we wanted. Let us make it more explicit: Again let τ be a uniformizer of K , and write $\text{Tr}(\pi^t) = a\tau^t$, $N(\pi^t) = b\tau^t$, with $a, b \in A$. Every element $x \in U_L^t/U_L^{t+1}$ is the image of an element of the form $1 + u\pi^t$ with $u \in A$. We compute:

$$\begin{aligned} N(1 + u\pi^t) &\equiv 1 + \text{Tr}(u\pi^t) + N(u\pi^t) \pmod{\mathfrak{m}_K^{t+1}} \\ &\equiv 1 + (bu + au^p)\tau^t \pmod{\mathfrak{m}_K^{t+1}}. \end{aligned}$$

Writing α, β for the images of a, b in k , we see that N_t can be written as the polynomial $\beta X + \alpha X^p$. As $\alpha \neq 0$, we see that the kernel of N_m has order at most $p = \ell$, but it contains $\theta_t(G_t)$, so it has to be equal to $\theta_t(G_t)$. Hence $\beta \neq 0$, and (c) is proved.

$m > t$: We computed in Example 3.50 that $\psi_{L/K}(m) = t + \ell(m - t)$. We have

$$\frac{(t+1)(\ell-1) + \psi_{L/K}(m)}{\ell} = \frac{\ell-1}{\ell} + m, \quad (3.8)$$

so $\text{Tr}(\mathfrak{m}_L^{\psi_{L/K}(m)}) \subseteq \mathfrak{m}_K^m$. Moreover, $\psi_{L/K}(m) \geq m+1$ and $N(\mathfrak{m}_L^{\psi_{L/K}(m)}) \subseteq \mathfrak{m}_K^{\psi_{L/K}(m)}$, so if $x \in \mathfrak{m}_L^{\psi_{L/K}(m)}$, then

$$N(1+x) \equiv 1 + \text{Tr}(x) \pmod{\mathfrak{m}_K^{m+1}}.$$

This shows that $N(U_L^{\psi_{L/K}(m)}) \subseteq U_K^m$ and similarly also $N(U_L^{\psi_{L/K}(m)+1}) \subseteq U_K^{m+1}$. Again, if $x = u\pi^r$ with $u \in A$, and $\text{Tr}(\pi^{\psi_{L/K}(m)}) = a_m \tau^m$, then $N(1 + u\pi^r) \equiv 1 + a_m u \tau^m \pmod{\mathfrak{m}_K^{m+1}}$, so N_m identifies with $\alpha_m X$, if α_m is the image of a_m in k . Finally, $\alpha_m \neq 0$, as otherwise $\text{Tr}(\mathfrak{m}_L^{\psi_{L/K}(m)}) \subseteq \mathfrak{m}_K^{m+1}$, which is impossible according to (3.8) and Lemma 3.58.

□

For future reference, we prove a fact about the norm of a cyclic extension.

Theorem 3.59 (Hilbert's Theorem 90). *Let L/K be a Galois extension of fields with cyclic Galois group G and generator g (no other assumptions are necessary). If $v \in L^\times$ has norm 1, then there exists $x \in L^\times$, such that $v = \frac{g(x)}{x}$. In other words, the quotient group*

$$\ker(N_{L/K}) / \left\{ \frac{g(x)}{x} \mid x \in L^\times \right\}$$

is trivial.

Proof. Write $V = \ker(N_{L/K})$ and $\overline{V} := V / \left\{ \frac{g(x)}{x} \mid x \in L^\times \right\}$. The group \overline{V} can be identified with the group cohomology $H^1(G, L^\times)$. A brief reminder: A crossed map $\varphi : G \rightarrow L^\times$, is a map φ , such that $\varphi(gh) = g(\varphi(h)) \cdot \varphi(g)$. A particular example of a crossed map is $g \mapsto \frac{gx}{x}$ for $x \in L^\times$; such a crossed map is called principal. The set of crossed maps inherits a group structure, and

$$H^1(G, L^\times) = \{\text{crossed maps}\} / \{\text{principal crossed maps}\}.$$

If G is a cyclic group with generator g and if $\varphi : G \rightarrow L^\times$ is a crossed map, then $\varphi(g) \in L^\times$ has norm 1. It is not difficult to check that this defines a homomorphism $u : \{\text{crossed maps}\} \rightarrow \ker(N_{L/K})$. Moreover, the map u is surjective: If $x \in L^\times$ has norm 1, then $\psi(1) := 1$, $\psi(g^r) := \prod_{i=0}^{r-1} g^i x$, $1 \leq r < |G|$, defines a crossed map with $\psi(g) = x$. Finally, one checks that

$$u^{-1} \left(\left\{ \frac{g(x)}{x} \mid x \in L^\times \right\} \right) = \{\text{principal crossed maps}\},$$

from which it follows that u induces an isomorphism $H^1(G, L^\times) \xrightarrow{\cong} \overline{V}$.

To prove the theorem, it hence remains to show that every crossed map $G \rightarrow L^\times$ is principal, and this is true even if $G = \text{Gal}(L/K)$ is not cyclic.

Let φ be a crossed map and for $y \in L^\times$ write

$$x := \sum_{\sigma \in G} \varphi(\sigma) \cdot \sigma(y).$$

If there exists y such that $x \neq 0$, then φ is principal. Indeed, in this case, for any $\tau \in G$ we can write

$$\tau(x) = \sum_{\sigma \in G} \tau(\varphi(\sigma)) \cdot \tau\sigma(y).$$

Since $\varphi(\tau\sigma) = \tau\varphi(\sigma) \cdot \varphi(\tau)$, we see that

$$\tau(x) = \varphi(\tau)^{-1} \sum_{\sigma \in G} \varphi(\tau\sigma) \tau\sigma(y) = \varphi(\tau)^{-1} \cdot x$$

which proves that φ is principal, if $x \neq 0$.

To see that we can find $y \in L^\times$ such that $x \neq 0$, one observes (this is not entirely trivial) that the family of maps $\sigma : L^\times \rightarrow L^\times$, $\sigma \in G$ is linearly independent over L . In particular: $\sum_{\sigma \in G} \varphi(\sigma) \cdot \sigma : L \rightarrow L$ is not the zero-map. □

Exercise 3.60. (a) Complete the proof that $H^1(G, L^\times) \cong V$.

- (b) If T is a group and $f_1, \dots, f_r : T \rightarrow L^\times$ pairwise distinct homomorphisms, prove that the f_i are linearly independent over K , i.e. if there are $a_1, \dots, a_r \in L$, such that $\sum_{i=1}^r a_i f_i = 0$ as map $T \rightarrow L$, then $a_1 = a_2 = \dots = a_r = 0$.

3.9. Jumps and the theorem of Hasse-Arf. We denote by L/K a finite Galois extension of complete discretely valued fields with separable residue extension. Let G be its Galois group. We know that there are only finitely many subgroups which appear in the ramification filtration, regardless of whether we use the upper or lower numbering. The real numbers $u \in [-1, \infty)$, where the upper numbering filtration $\{G^u\}$ transitions from one group to the next, are of special interest.

Definition 3.61. The numbers $u \in [-1, \infty)$, such that $G^u \not\supseteq G^{u+\varepsilon}$ for all $\varepsilon > 0$ are called *jumps* or *breaks* of this filtration.

The jumps are not necessarily integers, if L/K is not abelian.

Example 3.62. In [43, Sec. 4], an example of a totally ramified, finite Galois extension L/\mathbb{Q}_2 is constructed, for which the jumps of the upper numbering filtration of its Galois group are not integers. More precisely, the Galois group $G := \text{Gal}(L/\mathbb{Q}_2)$ is the group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ with $(-1)^2 = 1$, $i^2 = j^2 = k^2 = ijk = -1$. Its center is $Z(G) = \{\pm 1\}$, and L/\mathbb{Q}_2 has the ramification filtration $G_0 = G_1 = G$, $G_2 = G_3 = Z(G)$, $G_n = \{1\}$, $n > 3$. It follows that the two jumps are 1 and $3/2$.

Theorem 3.63 (Hasse-Arf). *If the group G is abelian, then the jumps are integers.*

Remark 3.64. We already know a few jumps of the ramification filtration: Since $\varphi_{L/K}(u) = u$ for $u \in [-1, 0]$, we see that the jumps in the filtration which happen in $[-1, 0]$ can only happen at -1 and 0 . Thus, in the proof of the Hasse-Arf theorem, it suffices to treat the jumps > 0 .

This deep theorem is fundamental for everything which we want to cover later on, so we will sketch a proof. For those who know local class field theory, the theorem of Hasse-Arf is easy to derive from it, at least in the case that the residue field of K is a finite field.

Example 3.65. Fix a separable closure K^{sep} of K , and let K^{ab} be the compositum of all finite abelian extensions of K contained in K^{sep} . If the residue field of K is finite, local class field theory states among other things that there exists a continuous morphism $\theta : K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$, the *local Artin map* or *local reciprocity map*, which becomes an isomorphism after profinite completion. Here, the profinite completion of the (non-discrete) topological group K^\times is to be understood as $\varprojlim K^\times/U$, where U runs through the finite index, normal, open subgroups.

More relevantly for us, if L/K is a finite abelian extension, then the composition

$$K^\times \xrightarrow{\theta} \text{Gal}(K^{\text{ab}}/K) \twoheadrightarrow \text{Gal}(L/K)$$

maps U_K^v onto $\text{Gal}(L/K)^v$ for all real $v \in [0, \infty)$, see [10, Thm. 1, Ch. VI.4]. Here we use the definition $U_K^v := U_K^{[v]}$. The jumps of the filtration $\{U_K^v\}$ are

obviously integers, hence the same is true for the upper numbering filtration of $\text{Gal}(L/K)$, whence the Hasse-Arf theorem follows from local class field theory, if the residue field of K is finite.

Of course proving the existence of the local reciprocity map and establishing its properties is difficult. We will follow [46] and give a more elementary proof of the Hasse-Arf theorem for arbitrary residue fields.

The proof of the Hasse-Arf Theorem 3.63 is fairly long and intricate, so we first make our life easier by simplifying the situation.

Proposition 3.66. *Recall that L/K is a finite Galois extension of complete discretely valued fields with separable residue extension. Additionally, assume that L/K is totally ramified and that the group $G = \text{Gal}(L/K)$ is cyclic.*

If μ is the largest integer such that $G_\mu \neq 1$, then $\varphi_{L/K}(\mu)$ is an integer.

This special case of Theorem 3.63 actually implies the theorem.

Lemma 3.67. *If Proposition 3.66 is true, then so is the Theorem 3.63 of Hasse-Arf.*

Proof. Let L/K be a finite abelian extension with Galois group G and let $v \in [0, \infty)$ be a jump, i.e. $G^v \not\supseteq G^{v+\varepsilon}$ for all $\varepsilon > 0$. By Remark 3.64 we may assume that $v > 0$, hence $G^v \subsetneq G^0 = G_0$. Replacing K by L^{G_0} , we may assume that L/K is totally ramified.

Let $\varepsilon_0 > 0$ be sufficiently small, so that $v + \varepsilon_0$ is smaller than the next jump in the filtration. Since G is a finite abelian group, so is $G/G^{v+\varepsilon_0}$, and the structure theorem for such groups tells us that there exists a cyclic group H , and a surjective map $\gamma : G \twoheadrightarrow H$, such that $\gamma(G^{v+\varepsilon_0}) = 1$, and $\gamma(G^v) \neq 1$.

The cyclic group H is the Galois group of a subextension L'/K of L/K , and by Proposition 3.53, we know that $H^u = \gamma(G^u)$ for all $u \in [-1, \infty)$. It follows that $H^{v+\varepsilon_0} = 1$ and $H^v \neq 1$, and that there are no jumps between v and $v + \varepsilon_0$. Finally, write $v = \varphi_{L'/K}(\mu)$. Since $H_\mu = H^{\varphi_{L'/K}(\mu)}$, by continuity of φ , it follows that $G_{\mu+\varepsilon} = 1$ for all $\varepsilon > 0$. This means that $\lfloor \mu \rfloor < \lfloor \mu + \varepsilon \rfloor$ for all $\varepsilon > 0$, and hence μ is an integer; in fact it is the largest integer such that $H_\mu \neq 1$. Now we are in the situation of Proposition 3.66, which implies, if true, that $v = \varphi_{L'/K}(\mu)$ is an integer. \square

To prove the Hasse-Arf Theorem 3.63, we can thus concentrate on totally ramified, cyclic extensions L/K , which we will do in the following sections.

Before we proceed with the proof of the Hasse-Arf Theorem, we continue Example 3.33.

Example 3.68. Let $K = k((x))$ with k an algebraically closed field of characteristic $p > 0$. Let L be the Galois extension given by $L := K[t]/(t^p - t - x^{-m})$, where $(m, p) = 1$; its Galois group is $G = \mathbb{F}_p$. We saw in Example 3.33 that $G_0 = G_1 = \dots = G_m \subsetneq G_{m+1} = 0$. Let us find the jumps of L/K . For this we just have to compute $\varphi_{L/K}(m)$, which is easy because of (3.6):

$$\varphi_{L/K}(m) = \sum_{i=0}^m \frac{|G_i|}{|G_0|} - 1 = m.$$

We see that in this simple example, the conclusion of the Hasse-Arf theorem holds true.

3.10. The ramification filtration of a cyclic extension. In this section, let L/K be a totally ramified, finite cyclic Galois extension of complete discretely valued fields, G its Galois group and g a generator. Let V be the kernel of the norm $N : L^\times \rightarrow K^\times$. By Hilbert's Theorem 90 (Theorem 3.59) $V = \{ \frac{gy}{y} | y \in L^\times \}$. Define

$$W := \left\{ x \in V \mid x = \frac{gy}{y} \text{ for some } y \in U_L \right\} \subseteq V,$$

where B is the valuation ring of L and $U_L = B^\times$. Clearly, W is a subgroup of V .

For $i \geq 0$, define $V_i := V \cap U_L^i$ and $W_i := W \cap U_L^i$. The quotients V_i/W_i define a filtration of the group V/W .

Proposition 3.69. *Let $\pi \in B$ be a uniformizer and consider G to be filtered by the ramification filtration $\{G_i\}$ in the lower numbering.*

- (a) *The assignment $\theta(\sigma) := \frac{\sigma\pi}{\pi}$ defines an isomorphism of groups $\theta : G \rightarrow V/W$.*
- (b) *θ respects the filtrations, i.e. $\theta|_{G_i} : G_i \hookrightarrow V_i/W_i$ for all $i \geq 0$.*

Proof. The following argument partly follows lecture notes by I. Fesenko. First, note that for any $\sigma \in G$, $\theta(\sigma) = \frac{\sigma(\pi)}{\pi} \in V$. Indeed, $v_L(\theta(\sigma)) = v_L(\sigma(\pi)) - v_L(\pi) = 0$, so $\theta(\sigma) \in U_L$. Also,

$$N_{L/K}(\theta(\sigma)) = \prod_{\tau \in G} \frac{\tau\sigma(\pi)}{\tau(\pi)} = \frac{\prod_{\tau \in G} \tau\sigma(\pi)}{\prod_{\tau \in G} \tau(\pi)} = 1.$$

Let us see that θ is a homomorphism of groups: For any $\tau, \sigma \in G$ we have

$$\frac{\tau\sigma(\pi)}{\pi} \cdot \frac{\pi^2}{\sigma(\pi)\tau(\pi)} = \frac{\sigma\left(\frac{\tau(\pi)}{\pi}\right)}{\frac{\tau(\pi)}{\pi}},$$

and $\frac{\tau(\pi)}{\pi} \in U_L$. Hence $\theta(\sigma\tau) \equiv \theta(\sigma)\theta(\tau) \pmod{W}$.

To see that θ is injective, assume that $\frac{g^r(\pi)}{\pi} \in W$. By the computation above, this means that there exists $u \in U_L$ such that $\left(\frac{g(\pi)}{\pi}\right)^r = \frac{g(u)}{u}$. But this means $g\left(\frac{\pi^r}{u}\right) = \frac{\pi^r}{u}$, so $\pi^r/u \in K$. Hence the degree $[L : K]$ divides r , so $g^r = 1$, and θ is injective.

For the surjectivity, note that every element in V has the shape $\frac{g(u\pi^i)}{u\pi^i}$, for $u \in U_L$ and $i \in \mathbb{Z}$. But in V/W we have $\frac{g(u\pi^i)}{u\pi^i} = \frac{g(\pi^i)}{\pi^i} = \theta(g^i)$.

Finally, if $\sigma \in G_i$, then $\frac{\sigma(\pi)}{\pi} \in U_L^i$ by definition, so θ indeed induces injective morphisms $\theta_i : G_i \rightarrow V_i/W_i$. \square

Proposition 3.70. *In addition to the assumptions and notations of Proposition 3.69, assume that the residue field of K is not the prime field \mathbb{F}_p . Then $G_m = 0$ if and only if $V_m/W_m = 0$.*

3.71 below.

Proof. We have seen that θ restricts to an injective map $G_m \hookrightarrow V_m/W_m$, so it just remains to show that $V_m/W_m = 0$ if $G_m = 0$. We proceed by descending induction on m . We know that $G_m = 0$ for $m \gg 0$, and the same is true for V_m/W_m . Indeed, as L/K is separable, there exists $t \in L$ such that

$\text{Tr}_{L/K}(t) = 1$. Write $M := \max\{0, -v_L(t)\}$. We claim that $V_m = W_m$ for all $m > M$. Let $x \in V_m$, and write

$$y := t + \sum_{i=1}^{|G|-1} xg(x)g^2(x) \cdots g^{i-1}(x)g^i(t).$$

Subtracting $\text{Tr}_{L/K}(t) = 1$ from both sides we obtain

$$y - 1 = \sum_{i=1}^{|G|-1} (xg(x)g^2(x) \cdots g^{i-1}(x) - 1)g^i(t),$$

and hence $v_L(y - 1) > 0$. It follows that $y \in U_L^1$, and that $\frac{y}{g(y)} = x$, since $N_{L/K}(x) = 1$:

$$\begin{aligned} \frac{y}{g(y)} &= \frac{xy}{xg(y)} \\ &= \frac{xy}{\sum_{i=1}^{|G|-1} xg(x) \cdots g^{i-1}(x)g^i(t) + N_{L/K}(x)t} \\ &= x. \end{aligned}$$

Thus $x = \frac{y}{g(y)} \in W_m$, so $W_m = V_m$ for $m > M$.

Now let $G_m = 0$, and assume that we know that $V_{m+1}/W_{m+1} = 0$. Our goal is to show that $V_m/W_m = 0$.

We will actually prove something stronger:

Lemma 3.71. *We continue to use the notations and assumptions from Proposition 3.70. In particular, the residue field of K is not the prime field. Write $n + 1 := \lceil \varphi_{L/K}(m) \rceil$. If $V_{m+1}/W_{m+1} = 0$ and $G_{\psi_{L/K}(n+1)} = 0$, then $V_m/W_m = 0$.*

Proof. Note that $\psi_{L/K}(n+1)$ is an integer $\geq m$ (Exercise 3.49), with equality if and only if $\varphi_{L/K}(m)$ is an integer.

We treat these two cases separately: $\varphi_{L/K}(m) \in \mathbb{Z}$ and $\varphi_{L/K}(m) \notin \mathbb{Z}$.

First, let us assume that $\varphi_{L/K}(m) \in \mathbb{Z}$, i.e. $\varphi_{L/K}(m) = n + 1$. Proposition 3.57 then produces a commutative diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & V_{m+1} & \longrightarrow & V_m & \longrightarrow & G_m/G_{m+1} \\ & & \downarrow & & \downarrow & & \downarrow \theta_m \\ 0 & \longrightarrow & U_L^{m+1} & \longrightarrow & U_L^m & \longrightarrow & U_L^m/U_L^{m+1} \longrightarrow 0 \\ & & \downarrow N & & \downarrow N & & \downarrow N_m \\ 0 & \longrightarrow & U_K^{n+2} & \longrightarrow & U_K^{n+1} & \longrightarrow & U_K^{n+1}/U_K^{n+2} \longrightarrow 0. \end{array}$$

As $V_{m+1}/W_{m+1} = 0$ by assumption, we see that the composition

$$V_m \rightarrow G_m/G_{m+1} \rightarrow (V_m/W_m)/(V_{m+1}/W_{m+1}) = V_m/W_m$$

is the canonical quotient map. But $G_m = G_{\psi_{L/K}(n+1)} = 0$, so $V_m/W_m = 0$.

Now assume that $\varphi_{L/K}(m) \notin \mathbb{Z}$, i.e.

$$n < \varphi_{L/K}(m) < n+1 \text{ and } \psi_{L/K}(n) + 1 \leq m < \psi_{L/K}(n+1).$$

We have $V_m/V_{m+1} \subseteq U_L^m/U_L^{m+1}$, and by assumption $V_{m+1} = W_{m+1}$. Thus, to show that $V_m/W_m = 0$, it suffices to show that V_m and W_m have the same image in U_L^m/U_L^{m+1} .

Let $x \in U_L^m$ be arbitrary. We have the inclusion $U_L^{\psi_{L/K}(n+1)} \subseteq U_L^m$, and the norm induces a surjective morphism $N : U_L^{\psi_{L/K}(n+1)} \twoheadrightarrow U_K^{n+1}$ by Proposition 3.57. It follows that there exists $y \in U_L^{\psi_{L/K}(n+1)}$ with $N(y) = N(x)$. Then $N(xy^{-1}) = 1$ and hence $xy^{-1} \in V_m$. As $\psi_{L/K}(n+1) \geq m+1$, we see that $y \in U_L^{m+1}$, so $x \equiv xy^{-1} \pmod{U_L^{m+1}}$. It follows that V_m maps surjectively to U_L^m/U_L^{m+1} .

As for the image of W_m , from what we showed in the previous paragraph, it follows that V_m/W_m maps surjectively to $(U_L^m/U_L^{m+1})/W_m$, so this group is also cyclic. But for $m > 0$, the group U_L^m/U_L^{m+1} is isomorphic to the additive group underlying the residue field of K , which is not cyclic, as the residue field of K is assumed not to be the prime field. Thus the image of W_m in U_L^m/U_L^{m+1} is nontrivial.

Let $x \in W_m \setminus U_L^{m+1}$. Then there exists $y \in U_L$ such that $x = \frac{g(y)}{y}$. By modifying y with an element from U_K , we may assume that $y \in U_L^1$, i.e. $y = 1 + z$, $v_L(z) > 0$. For $a \in U_K$ define $y_a := 1 + az$ and $x_a := \frac{g(y_a)}{y_a}$.

Denote by $\gamma : U_L^{m+1}/U_L^m \rightarrow \mathfrak{m}_L^m/\mathfrak{m}_L^{m+1}$ the isomorphism given by $\gamma(1 + u\pi^m) = u\pi^m \pmod{\mathfrak{m}_L^{m+1}}$, where π is a uniformizer of L and \mathfrak{m}_L the maximal ideal of the discrete valuation ring of L . We claim that $x_a \in W_m$, and that $\gamma(x_a) = a\gamma(x) \pmod{\mathfrak{m}_L^{m+1}}$. Since $\mathfrak{m}_L^m/\mathfrak{m}_L^{m+1}$ is a 1-dimensional vector space over the residue field of K , this would show that W_m maps surjectively to U_L^m/U_L^{m+1} , which is what we wanted to show.

As for the claim, we compute

$$\begin{aligned} x_a - 1 &= \frac{g(y_a) - y_a}{y_a} \\ &= \frac{1 + ag(z) - (1 + az)}{y_a} \\ &= \frac{a(gz - z)}{y_a} \\ &= a \frac{g(y) - y}{y_a} \\ &= a \frac{y}{y_a} (x - 1). \end{aligned}$$

□

It follows that $v_L(x_a - 1) = v_L(a) + m \geq m$, so $x_a \in W_m$. Moreover, if $x - 1 = u\pi^m$ and $\frac{y}{y_a} = 1 + v\pi$ for some $u \in U_L$ and $v \in B$, then

$$x_a - 1 = au\pi^m + avu\pi^{m+1},$$

so $\gamma(x_a) = a\gamma(x) \pmod{\mathfrak{m}_L^{m+1}}$, as claimed. This completes the proof of the lemma. \square

Remark 3.72. If the residue field of K is algebraically closed, almost the same proof shows that the isomorphism $\theta : G \xrightarrow{\cong} V/W$ induces isomorphisms $G_i \xrightarrow{\cong} V_i/W_i$ for all i .

Finally we can prove Proposition 3.66 and hence the Hasse-Arf theorem 3.63.

Proof of Proposition 3.66. Recall that L/K is a totally ramified, cyclic extension of complete discretely valued fields with separable residue extension. If the residue field of K is the prime field, we can replace K by an unramified extension with larger residue field. This does not change G_i for $i \geq 0$ (see [46, §5]). If μ is the largest positive integer such that $G_\mu \neq 0$, then we want to show that $\varphi_{L/K}(\mu)$ is an integer. If it is not an integer, then there exists an integer ν , such that $\nu < \varphi_{L/K}(\mu) < \nu + 1$, and hence $\mu < \psi_{L/K}(\nu + 1)$. It follows that $G_{\psi_{L/K}(\nu+1)} = 0$. Now Lemma 3.71 applies, and thus $V_\mu/W_\mu = 0$. But we have seen in Proposition 3.69 that $\theta : G \xrightarrow{\cong} V/W$ respects the filtrations, so $G_\mu \hookrightarrow V_\mu/W_\mu = 0$, which is a contradiction. Thus $\varphi_{L/K}(\mu) \in \mathbb{Z}$, and the proof is complete. \square

4. THE SWAN REPRESENTATION

Let K be a complete discretely valued field with perfect residue field and K^{sep} a separable closure. If ℓ is a prime number different from the residue characteristic of K and E a finite extension of \mathbb{Q}_ℓ , then we want to study continuous morphisms $\rho : \text{Gal}(K^{\text{sep}}/K) \rightarrow \text{GL}_r(E)$, where $\text{GL}_r(E)$ is considered as a topological group with the topology induced by E (see Definition 4.65).

In Section 4.4 will define the *Swan conductor* $\text{sw}(\rho) \in \mathbb{Z}$ of such a representation, which is additive in short exact sequences and zero if and only if ρ is *tame*, i.e. if ρ restricts to the trivial representation on the wild ramification group. We will see in Section 9 that this invariant has a cohomological interpretation. Before we come to the Swan conductor, we will use the theorem of Hasse-Arf to attach to a finite Galois extension L/K with group G , first a complex representation, then a projective $\mathbb{Z}_\ell[G]$ -module Sw_G , the *Swan representation*.

4.1. A minimal amount of representation theory of finite groups. We roughly follow [45], but [30] and [49] have also been helpful.

4.1.1. Basics. Let G be a finite group, E a field and V a finite dimensional E -vector space. A morphism $\rho : G \rightarrow \text{GL}(V)$ is called a *representation of E* . If $\rho' : G \rightarrow \text{GL}(V')$ is a second representation, then a *morphism of representations* $\alpha : \rho \rightarrow \rho'$ is an E -linear map $\varphi : V \rightarrow V'$, such that for every $g \in G$, $v \in V$, we have $\varphi(\rho(g)(v)) = \rho'(g)(\varphi(v))$. We write $\text{Rep}_E G$ for the category of representations of G on finite dimensional E -vector spaces. This category is equivalent to the category of finitely generated left modules over the group ring $E[G]$. Recall that $E[G] = E \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, where $\mathbb{Z}[G]$ is the free \mathbb{Z} -module with basis G , and multiplication induced by the rule $ag \cdot bg' =$

$(ab)gg' \in \mathbb{Z}[G]$, for $a, b \in \mathbb{Z}$. From this perspective it is clear that we have the usual notions of subrepresentations, quotients and constructions like direct sums, products, etc. In the sequel, we will use the notions of “representation” and “finitely generated $E[G]$ -module” interchangeably. Unless explicitly stated otherwise, the word “ $E[G]$ -module” means “finitely generated left- $E[G]$ -module”.

Let $\varphi : H \rightarrow G$ be a homomorphism of groups, and $\psi : E[H] \rightarrow E[G]$ the induced morphism of E -algebras. If V is a left- $E[G]$ -module, then using ψ we can also consider it as an $E[H]$ -module. The corresponding representation is called the *restriction of V to H* , denoted $\text{Res}_\varphi V$. Similarly, if W is an $E[H]$ -module, then $E[G] \otimes_{E[H]} W$ is an $E[G]$ -module. The corresponding representation is said to be the *representation induced by φ and W* , denoted $\text{Ind}_\varphi W$. These constructions define functors $\text{Ind}_\varphi : \text{Repf}_E H \rightarrow \text{Repf}_E G$, and $\text{Res}_\varphi : \text{Repf}_E G \rightarrow \text{Repf}_E H$. Considering the module theoretic definitions of these constructions, we see that these functors are actually adjoint:

$$\text{Hom}_{\text{Repf}_E H}(-, \text{Res}_\varphi(-)) = \text{Hom}_{\text{Repf}_E G}(\text{Ind}_\varphi(-), -). \quad (4.1)$$

Here are some important examples of representations.

Example 4.1.

- (a) The 1-dimensional vector space E with G -action given by $ge = e$ for all $g \in G$, $e \in E$. This is the *trivial representation of rank 1*, which we denote by $\mathbf{1}_G$ or just E . If $\rho : G \rightarrow \text{GL}(V)$ is a finite dimensional E -representation of G , such that $\rho(g) = \text{id}$ for all $g \in G$, then ρ is isomorphic to a direct sum of finitely many copies of $\mathbf{1}_G$. Such a representation ρ is called *trivial*. (The trivial $E[G]$ -module is the trivial G -representation of rank 0.)
If V is an $E[G]$ -module, we write $V^G := \{v \in V \mid \forall g \in G : gv = v\}$. This is the maximal trivial subrepresentation of V .
- (b) The ring $E[G]$ is a left- $E[G]$ -module. The corresponding representation is called the *regular representation* of G over E .
- (c) We have a surjective morphism of $E[G]$ -modules $E[G] \rightarrow E$, $g \mapsto 1$. In other words: The trivial representation of rank 1 is a quotient of the regular representation. Its kernel is the ideal $\text{Aug}_E(G) := (g - 1 \mid g \in G)$, and it is called the *augmentation representation*.
- (d) If the order of G is invertible in E , then the short exact sequence

$$0 \rightarrow \text{Aug}_E(G) \rightarrow E[G] \rightarrow E \rightarrow 0$$

is split by the map $\theta : E \rightarrow E[G]$, $\theta(e) = \frac{e}{|G|} \sum_{g \in G} g$.

- (e) If V, W are two finite dimensional representations of G over E , then $\text{Hom}_E(V, W)$ has a canonical structure of a representation of G : If $\varphi \in \text{Hom}_E(V, W)$ and $g \in G$, then define $g \bullet \varphi$ by $(g \bullet \varphi)(v) = g\varphi(g^{-1}v)$. Clearly $\text{Hom}_E(V, W)^G = \text{Hom}_{E[G]}(V, W)$.
We write $\underline{\text{Hom}}(V, W)$ for the representation on $\text{Hom}_E(V, W)$ just defined.
- (f) If V is a finite dimensional representation of G , then $\underline{\text{Hom}}(V, \mathbf{1}_G)$ is a finite dimensional representation of G , the *dual representation* V^\vee of V . The underlying vector space is the dual vector space $\text{Hom}_E(V, E)$ of V and G acts via $(g \bullet \varphi)(v) = \varphi(g^{-1}v)$.

- (g) If V and W are finite dimensional representations of G , then the tensor product $V \otimes_E W$ carries a canonical G -action, defined via $g(v \otimes w) = gv \otimes gw$ for $g \in G$, $v \in V$, $w \in W$. Using this definition we see that the natural isomorphism of E -vector spaces $V^\vee \otimes_E W \rightarrow \text{Hom}_E(V, W)$ is G -invariant, and hence induces a natural isomorphism of representations

$$V^\vee \otimes W \cong \underline{\text{Hom}}(V, W).$$

Similarly we see that if U is a third representation of G , then there is a natural isomorphism

$$\underline{\text{Hom}}(U \otimes V, W) \cong \underline{\text{Hom}}(U, \underline{\text{Hom}}(V, W)).$$

Definition 4.2. A representation V of G over E is called

- *irreducible* or *simple* if it is irreducible as $E[G]$ -module, i.e. if it does not have nontrivial sub- $E[G]$ -modules.
- *semi-simple* if V is a direct sum of irreducible $E[G]$ -modules.
- *indecomposable* if it is not isomorphic to a direct sum $V_1 \oplus V_2$ with V_1, V_2 nonzero representations.

Theorem 4.3 (Maschke). *If G is a finite group with order coprime to $\text{char}(E)$, then every representation of G on a finite dimensional E -vector space is semi-simple.*

Proof. Let V be a finite dimensional E -representation of G , and $W \subseteq V$ a subrepresentation. Pick a projection $P : V \rightarrow W$ of E -vector spaces, which splits the inclusion $W \subseteq V$, i.e. such that $P|_W = \text{id}_W$, and define

$$P_0 := \frac{1}{|G|} \sum_{g \in G} gPg^{-1}.$$

It is easy to check that P_0 is a morphism of $E[G]$ -modules, and we still have $P_0|_W = \text{id}_W$. It follows that $\ker(P_0)$ is a subrepresentation of V , and $V = W \oplus \ker(P_0)$ as $E[G]$ -modules. \square

To see the contrast to the situation where $|G|$ and $\text{char}(E)$ are not coprime, do the following exercise.

Exercise 4.4. Let $p = \text{char}(E)$. If G is a finite group, then the two cases $(|G|, p) = 1$ and $(p, |G|) > 0$ are very different.

As a “worst case” scenario, prove that if $p > 0$ and if G is a p -group acting on a finite dimensional E -vector space $V \neq 0$, then $V^G \neq 0$. Here $V^G = \{v \in V \mid \forall g \in G : gv = v\}$.

Deduce that V is a successive extension of the trivial rank 1 representation of G by itself.

Proposition 4.5. *Let E be a field containing all roots of unity of order dividing $|G|$. If G is abelian, then every irreducible representation of G over E has rank 1. If $|G|$ is invertible in E , then the converse is also true.*

Proof. Let $\rho : G \rightarrow \text{GL}(V)$ be a nonzero irreducible representation of G . For $g \in G$, the order of $\rho(g)$ divides $|G|$. Let $v \in V$ be an eigenvector of $\rho(g)$ and λ its eigenvalue. Then λ is a root of unity of order dividing $|G|$, thus $\lambda \in E$ by assumption. For any other $h \in G$ we have $\rho(gh)(v) = \rho(hg)(v) = \lambda\rho(h)(v)$.

Thus the λ -eigenspace of $\rho(g)$ is a nonzero subrepresentation of ρ , hence equal to ρ as ρ is irreducible. This shows that every element $g \in G$ acts via scalar multiplication on V . As ρ is irreducible, $\dim V = 1$.

If $|G|$ is invertible in E , let ρ be a faithful representation of G , i.e. an injective map $\rho : G \rightarrow \mathrm{GL}(V)$ (such a representation always exists). If the irreducible representations of G have rank 1, Maschke's Theorem shows that the matrices $\{\rho(g) | g \in G\}$ are simultaneously diagonalizable. It follows that G is abelian. \square

It is convenient to introduce the following notation.

Definition 4.6. If G is a finite group and E a field, we write $R_E(G)$ for the *Grothendieck ring* of the category $\mathrm{Rep}_E G$. Recall that $R_E(G)$ is the abelian group generated by isomorphism classes $[V]$ of finite dimensional E -representations of G , subject to the relations $[V] = [V_1] + [V_2]$ if there exists a short exact sequence

$$0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0.$$

The tensor product over E makes $R_E(G)$ into a commutative ring (*Exercise: Check this!*). It is easy to see that $R_E(G)$ is the free \mathbb{Z} -module generated by the isomorphism classes of irreducible representations of G . We will see in Proposition 4.7 that there are only finitely many of those.

There is a natural map of monoids

$$(\{\text{isomorphism classes of objects in } \mathrm{Rep}_E G\}, \oplus) \rightarrow R_E(G). \quad (4.2)$$

Its image is the subset of all linear combinations $\sum_{V_i \text{ irreducible}} a_i [V_i]$, $a_i \geq 0$, and is denoted by $R_E^+(G)$.

Maschke's Theorem 4.3 can be phrased as follows: If the order $|G|$ is prime to the characteristic of E , then the map (4.2) is injective.

Next, we see that there are only finitely many irreducible representations of a finite group.

Proposition 4.7. *If G is a group and E a field, then any irreducible finite dimensional representation of G on E is a composition factor of the regular representation $E[G]$. In particular, up to isomorphism there are only finitely many irreducible representations of G on E .*

Proof. If V is an irreducible $E[G]$ -module, pick any $v \in V \setminus \{0\}$. The map $x \mapsto xv$, is a $E[G]$ -linear morphism $E[G] \rightarrow V$, which is surjective as V is irreducible. The claim follows. \square

We close this section with a fundamental but easy lemma.

Lemma 4.8 (Schur's Lemma). *Let V_1, V_2 be two finite dimensional irreducible $E[G]$ -modules. If $f : V_1 \rightarrow V_2$ is a morphism of representations, then precisely one of the following statements is true:*

- (a) $f = 0$.
- (b) f is an isomorphism and if E is algebraically closed, then for any other isomorphism $g : V_1 \xrightarrow{\cong} V_2$, there exists $a \in E$, such that $f = ag$.

In particular, if V is an irreducible $E[G]$ -module, then $\mathrm{End}_{E[G]}(V)$ is a division algebra, and $\mathrm{End}_{E[G]}(V) = E$ if E is algebraically closed.

Proof. If f is not an isomorphism, then $f = 0$, due to the irreducibility of V_1, V_2 . Now assume that f and g are isomorphisms of $E[G]$ -modules and that E is algebraically closed. Let a be an eigenvalue of $g^{-1}f : V_1 \rightarrow V_2$. Then $g^{-1}f - a \text{id}_{V_1}$ has a nonzero kernel, so $g^{-1}f = a \cdot \text{id}_{V_1}$ as claimed. \square

We give an exemplary application:

Proposition 4.9. *Let G be a group and E a field such that $|G|$ is invertible in E . If V is a representation of G such that $\text{End}_{E[G]}(V) = E$, then V is irreducible. If E is algebraically closed, the converse is true as well.*

Proof. Since $|G|$ is invertible V can be written as $V = V_1 \oplus V_2$ with V_1 irreducible. We compute

$$\dim_E \text{End}_{E[G]}(V) \geq \dim_E \text{End}_{E[G]}(V_1) + \dim_E \text{End}_{E[G]}(V_2) \geq 2$$

if $V_2 \neq 0$. It follows that $V = V_1$ if $\text{End}_{E[G]}(V) = E$. The converse is just Schur's Lemma. \square

4.1.2. *Extension of scalars.* If $E \subseteq F$ is an extension of fields, then $-\otimes_E F$ gives rise to a functor $\text{Rep}_E(G) \rightarrow \text{Rep}_F(G)$. In this section, we analyze this functor.

Lemma 4.10. *Let $E \subseteq F$ be an extension of fields and G a finite group. If V is an $E[G]$ -module, then $V \otimes_E F = F[G] \otimes_{E[G]} V$ is an $F[G]$ -module. The natural map*

$$V^G \otimes_E F \rightarrow (V \otimes_E F)^G$$

is an isomorphism. Consequently, if W is a second $E[G]$ -module, the natural map

$$\text{Hom}_{E[G]}(V, W) \otimes_E F \cong \text{Hom}_{F[G]}(V \otimes_E F, W \otimes_E F)$$

is an isomorphism.

Proof. Since F is flat over E , the natural map of F -vector spaces

$$\text{Hom}_E(V, W) \otimes_E F \rightarrow \text{Hom}_F(V \otimes_E F, W \otimes_E F),$$

is an isomorphism. It is easily seen to be G -equivariant, so $\underline{\text{Hom}}(V, W) \otimes_E F \cong \underline{\text{Hom}}(V \otimes_E F, W \otimes_E F)$ as $F[G]$ -modules, and the second statement of the lemma follows from the first as $\text{Hom}_{E[G]}(V, W) = \underline{\text{Hom}}(V, W)^G$.

For the first statement, consider the E -linear map $\varphi : V \rightarrow \bigoplus_{g \in G} V$, $v \mapsto (gv - v)_{g \in G}$. Its kernel is the E -vector space V^G . Tensoring with F gives the map

$$\varphi \otimes \text{id}_F : V \otimes_E F \rightarrow (V \otimes_E F)^{|G|}, \quad v \otimes \lambda \mapsto (g(v \otimes \lambda) - v \otimes \lambda)_{g \in G}$$

Its kernel is $(V \otimes_E F)^G$, but it is also $V^G \otimes_E F$, as F is flat over E . \square

Proposition 4.11. *Let $E \subseteq F$ be an arbitrary field and G a finite group. The ring homomorphism $\varphi : R_E(G) \rightarrow R_F(G)$ induced by $-\otimes_E F$ is injective.*

Proof. Recall that $R_E(G)$ (resp. $R_F(G)$) is the free abelian group generated by the classes of irreducible $E[G]$ -modules (resp. classes of irreducible $F[G]$ -modules). If $c \in R_E(G)$ we can write $c = c_+ - c_-$ with $c_+, c_- \in R_E^+(G)$, where $R_E^+(G)$ is the submonoid of $\mathbb{Z}_{\geq 0}$ -linear combinations of classes of irreducible $E[G]$ -modules. We have $\varphi(c) = 0$ if and only if $\varphi(c_+) = \varphi(c_-)$. Thus it suffices to show that the map $R_E^+(G) \rightarrow R_F^+(G)$ is injective.

Let V_1, V_2 be irreducible finitely generated $E[G]$ -modules. If $\text{char}(E) = 0$, then by Maschke's Theorem 4.3 the $F[G]$ -modules $V_1 \otimes_E F$ and $V_2 \otimes_E F$ are semi-simple, i.e. direct sums of irreducible $F[G]$ -modules.

If $V_1 \not\cong V_2$, then the decompositions over F have no common factors. Indeed, otherwise Lemma 4.10 would imply that

$$0 = \dim_E \text{Hom}_{E[G]}(V_1, V_2) = \dim_F \text{Hom}_{F[G]}(V_1 \otimes_E F, V_2 \otimes_E F) > 0.$$

This shows that $R_E^+(G) \rightarrow R_F^+(G)$ is injective, if $\text{char}(E) = 0$.

If $\text{char}(E) > 0$, we have to generalize the argument a little ([30, (7.13)]): If $V_1 \not\cong V_2$ are irreducible, then $V_1 \otimes_E F$ and $V_2 \otimes_E F$ are perhaps not semi-simple, but we show that their composition series have no common factor. To this end, we show that there is an element $e \in E[G]$ acting as the identity on V_1 and as 0 on V_2 . Then $e \otimes 1$ acts as identity on all composition factors of $V_1 \otimes_E F$ and as 0 on all composition factors of $V_2 \otimes_E F$, which would complete the proof of the proposition.

Every simple $E[G]$ -module is of the shape $E[G]/M$, where $M \subseteq E[G]$ is a maximal left-ideal. Indeed, if V is a simple $E[G]$ -module, pick $v \in V \setminus \{0\}$. The map $\varphi_v : E[G] \rightarrow V, x \mapsto xv$ is surjective and its kernel is maximal, as V is simple. Let $\text{rad}(E[G])$ denote the intersection of all maximal left-ideals. If $x \in \text{rad}(E[G])$, and if V is a simple $E[G]$ -module, then for any $v \in V$, $x \in \ker(\varphi_v)$, so x acts trivially on every simple $E[G]$ -module. Conversely, if $x \in E[G]$ acts trivially on every simple $E[G]$ -module, then $x \in \text{rad}(E[G])$, as for every maximal left-ideal M , x must map to 0 in the simple module $E[G]/M$.

Thus $\text{rad}(E[G])$ is the set of elements acting trivially on every simple $E[G]$ -module. This shows that $\text{rad}(E[G])$ is a two-sided ideal, so that $E[G]/\text{rad}(E[G])$ is a ring. As a left- $E[G]$ -module, $E[G]/\text{rad}(E[G])$ is semi-simple; in fact it is the direct sum of all simple $E[G]$ -modules, with multiplicities (see Lemma 4.56). If, say,

$$E[G]/\text{rad}(E[G]) = V_1 \oplus \dots \oplus V_n$$

with V_1, \dots, V_n simple $E[G]$ -modules (hence left-ideals of $E[G]/\text{rad}(E[G])$), then there are orthogonal idempotents $e_1, \dots, e_n \in E[G]/\text{rad}(E[G])$, such that $1 = e_1 + \dots + e_n$, and such that for every $j = 1, \dots, n$, e_j acts as the identity on V_j , but as 0 on all V_i with $i \neq j$. If $f_j \in E[G]$ is a lift of e_j , then f_j acts as identity on V_j and as 0 on $V_i, i \neq j$. The proof is complete. \square

Definition 4.12. Let G be a finite group and E a field. We say that a representation V of G on E is *absolutely irreducible* if $V \otimes_E F$ is irreducible for all fields $F \supseteq E$.

If $|G|$ is invertible in E , then we can easily give a criterion for absolute irreducibility:

Proposition 4.13. *Let G be a group and E a field such that $|G|$ is invertible in E . The following statements for a representation V of G on E are equivalent:*

- (a) $\text{End}_{E[G]}(V) = E$
- (b) V is absolutely irreducible.
- (c) There exists an algebraically closed field $F \supseteq E$ such that $V \otimes_E F$ is irreducible.

Proof. Assume (a) and let F be an extension of E . Then $\text{End}_{F[G]}(V \otimes_E F) = F$ by Lemma 4.10, so $V \otimes_E F$ is irreducible according to Proposition 4.9, so V is absolutely irreducible.

(b) trivially implies (c). Assume (c) holds. Then $\dim_F \text{End}_{F[G]}(V \otimes_E F) = 1$ according to Proposition 4.9, so again we see that $\text{End}_{E[G]}(V) = E$. This completes the proof. \square

Remark 4.14. An analogous criterion is true even if $|G|$ is not invertible in E , but the proof is more complicated, see e.g. [30, Thm. 7.5].

Definition 4.15. We say that E is a *splitting field* for G , if every irreducible representation of G over E is absolutely irreducible.

Proposition 4.16. *If G is a finite group and E a splitting field, then any field $F \supseteq E$ is a splitting field. In this case the map $R_E(G) \rightarrow R_F(G)$ is an isomorphism.*

Proof. Assume that E is a splitting field. If $0 \subsetneq C_1 \subsetneq \dots \subsetneq C_n = E[G]$ is a composition series, then

$$0 \subsetneq C_1 \otimes_E F \subsetneq \dots \subsetneq C_n \otimes_E F = F[G]$$

is a composition series of $F[G]$, as $(C_i \otimes_E F)/(C_{i-1} \otimes_E F) = (C_i/C_{i-1}) \otimes_E F$ is (absolutely) irreducible for $i = 1, \dots, n$ by assumption. If W is an irreducible $F[G]$ -module, then $W \cong (C_i/C_{i-1}) \otimes_E F$ for some i according to Proposition 4.7, so W is absolutely irreducible and F a splitting field. Moreover, the class of W in $R_F(G)$ lies in the image of the map $R_E(G) \hookrightarrow R_F(G)$. \square

Proposition 4.13 translates to:

Corollary 4.17. *Let E be a field and G a group such that $|G|$ is invertible in E . If E is algebraically closed, then E is a splitting field.*

Remark 4.18. The same statement is true if $|G|$ is not necessarily invertible, but the proof is a little bit more complicated, see [30, Thm. 8.3].

Proposition 4.19. *Let G be a finite group and E a field. If E is a splitting field (e.g. algebraically closed), then every irreducible representation of G on E comes via base change from a finite extension E_0 of the prime field.*

Proof. Write \mathbb{F} for the prime field of E . Let \overline{E} be an algebraic closure of E and V an irreducible representation of G over E . If E is a splitting field, then $V \otimes \overline{E}$ is also irreducible, so it comes via base change from $\overline{\mathbb{F}} \subseteq \overline{E}$ according to Proposition 4.16, and hence from a finite extension $\mathbb{F} \subseteq E_V$ contained in $\overline{\mathbb{F}}$. Let E_0 be the compositum of the fields E_V in $\overline{\mathbb{F}}$. As there are only finitely many isomorphism classes of irreducible representations of G (Proposition 4.7), E_0 is finitely generated over \mathbb{F} . \square

Remark 4.20. If $\text{char}(E) = 0$, we will see in Corollary 4.43 that one can always take $E_0 = \mathbb{Q}(\zeta_m)$ as a splitting field, where m is the exponent of G and ζ_m an m -th root of unity. Moreover, the functor $\text{Rep}_{\mathbb{Q}(\zeta_m)}(G) \rightarrow \text{Rep}_{\overline{E}}(G)$ is an equivalence because of Maschke's Theorem and Lemma 4.10.

If $\text{char}(E) = p > 0$, then one can show that the field $\mathbb{Z}[\zeta_m]/\mathfrak{p}$ is a splitting field for G , where \mathfrak{p} is any prime ideal of $\mathbb{Z}[\zeta_m]$ lying over p .

4.1.3. *Characters.* We continue to denote by E a field and by G a finite group.

Definition 4.21. Let G be a finite group. A map $\varphi : G \rightarrow E$ satisfying the condition

$$\varphi(hgh^{-1}) = \varphi(g)$$

for all $g, h \in G$, is called *class function*.

If $\rho : G \rightarrow \text{GL}(V)$ is a representation of G on a finite dimensional E -vector space, then $\chi_\rho : G \rightarrow E$, $\chi_\rho(g) := \chi_V(g) := \text{Tr}(\rho(g))$ is a class function; it is called the *character of ρ* .

If ρ is irreducible, then its character is also called *irreducible*.

Lemma 4.22. If $\rho : G \rightarrow \text{GL}(V)$ and $\rho' : G \rightarrow \text{GL}(V')$ are isomorphic representations over E of the finite group G , then $\chi_\rho = \chi_{\rho'}$.

Proof. If $\gamma : V \rightarrow V'$ is a G -invariant E -isomorphism, then $\rho(g) = \gamma\rho'(g)\gamma^{-1}$, and $\text{Tr}(\rho(g)) = \text{Tr}(\gamma\rho'(g)\gamma^{-1})$. \square

Example 4.23. (a) Let V_1, V_2 be two $E[G]$ -modules with characters χ_1, χ_2 .

- If V is an $E[G]$ -module with character χ , which is an extension of V_1 by V_2 , then $\chi = \chi_1 + \chi_2$.
- The character of $V_1 \otimes V_2$ is $\chi_1 \cdot \chi_2$.
- If χ_1^\vee is the character of the dual representation V_1^\vee , then we see directly from the definition that $\chi_1^\vee(g) = \overline{\chi_1(g^{-1})}$ for all $g \in G$. If E is a subfield of \mathbb{C} , then $\chi_1^\vee(g) = \overline{\chi_1(g)}$, where $\overline{(-)}$ denotes complex conjugation. Indeed, the eigenvalues of each g acting on V_1 are roots of unity, and if $\xi \in \mathbb{C}$ is a root of unity, then $\xi^{-1} = \overline{\xi}$.
- The representation $\text{Hom}(V_1, V_2)$ has the character $\chi_2 \cdot \chi_1^\vee$, so if $E \subseteq \mathbb{C}$, then we can write $\chi_2 \cdot \overline{\chi_1}$.

- (b) Let E be the trivial rank 1 representation. Its character is the constant map $g \mapsto 1 \in E$. We denote it by 1_G .
- (c) Let r_G denote the character of the regular representation $E[G]$ of G . Then $r_G(1) = |G|$, and $r_G(g) = 0$ for $g \neq 1$. Indeed, if $g \neq 1$, then $gh \neq h$ for all h , so the $|G| \times |G|$ -matrix of g acting on $E[G]$ (with respect to standard basis) has only zeroes on the diagonal.
- (d) From the previous two examples we can compute the character of the augmentation representation (Example 4.1), which we denote by u_G : $u_G = r_G - 1_G$, so $u_G(1) = |G| - 1$ and $u_G(g) = -1$ for $g \neq 1$.

Definition 4.24. Assume that $|G|$ is invertible in E . Let G be a finite group and write $\mathbf{C}_{E,G}$ for the E -vector space of all class functions $G \rightarrow E$. For arbitrary maps $\varphi, \psi : G \rightarrow E$ we define

$$\langle \varphi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \psi(g^{-1}).$$

This is a symmetric bilinear form on the space of maps $G \rightarrow E$, and also on $\mathbf{C}_{E,G}$. If confusion is unlikely, we drop the subscript G from $\langle - | - \rangle_G$.

Proposition 4.25 (Orthogonality of characters). *As before, assume that $|G|$ is invertible in E .*

(a) If V_1, V_2 are arbitrary representations with characters χ_1, χ_2 , then

$$\langle \chi_1, \chi_2 \rangle = \dim_E(\text{Hom}_{E[G]}(V_1, V_2)).$$

(b) Let χ_1, χ_2 characters of irreducible representations V_1, V_2 of G . Then

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} a \in \mathbb{Z}_{\geq 1} & \text{if } V_1 \cong V_2 \\ 0 & \text{otherwise.} \end{cases}$$

(c) If E is algebraically closed then $a = 1$.

Proof. Taking Schur's Lemma 4.8 into account, it follows that we just have to prove (a). First let V be any representation and write $V = V/V^G \oplus V^G$. The element $\frac{1}{|G|} \sum_{g \in G} g \in E[G]$ induces an $E[G]$ -linear map $V \rightarrow V$ whose image is V^G , and which is the identity restricted to V^G . It follows that

$$\text{Tr} \left(\frac{1}{|G|} \sum_{g \in G} g \right) = \dim_E V^G.$$

We apply this observation to $\underline{\text{Hom}}(V_1, V_2)$. The character of $\underline{\text{Hom}}(V_1, V_2)$ is $\chi_{V_1}^\vee \cdot \chi_{V_2}$, so

$$\begin{aligned} \dim_E \text{Hom}_{E[G]}(V_1, V_2) &= \dim_E (\underline{\text{Hom}}(V_1, V_2))^G \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{V_1}(g^{-1}) \chi_{V_2}(g) \\ &= \langle \chi_{V_1}, \chi_{V_2} \rangle. \end{aligned}$$

□

Corollary 4.26. Assume that $|G|$ is invertible in E . The isomorphism class of a representation over E is determined by its character.

More precisely, if V is an E -representation of G , then

$$V \cong \bigoplus_{W \text{ irreducible}} W^{\frac{\langle \chi_V, \chi_W \rangle}{\langle \chi_W, \chi_W \rangle}}$$

where the direct sum is over the distinct irreducible representations of G on E .

Proposition 4.13 translates to:

Corollary 4.27. As before, assume that $|G|$ is invertible in E . If V is a representation of G over E , then V is absolutely irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Definition 4.28. We say that an irreducible representation W is contained in V with multiplicity m , if $\frac{\langle \chi_V, \chi_W \rangle}{\langle \chi_W, \chi_W \rangle} = m$, i.e. if m is maximal such that $W^{\oplus m} \subseteq V$.

Corollary 4.29. If G is a finite group and $|G|$ invertible in E , then every irreducible representation of G on an d -dimensional E -vector space is contained in the regular representation $E[G]$ with multiplicity $d / \langle \chi_W, \chi_W \rangle$.

If W_1, \dots, W_r are the distinct irreducible representations of G , then

$$|G| = \sum_{i=1}^r \frac{(\dim W_i)^2}{\langle \chi_{W_i}, \chi_{W_i} \rangle}.$$

Proof. If W is an irreducible representation of G , then by Example 4.23 we can compute

$$\langle r_G, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} r_G(g) \chi_W(g^{-1}) = \chi_W(1) = \dim W.$$

It follows that

$$E[G] = \sum_{i=1}^r W_i^{\frac{\dim W_i}{\langle \chi_{W_i}, \chi_{W_i} \rangle}},$$

so

$$|G| = \dim_E E[G] = \sum_{i=1}^r \frac{(\dim W_i)^2}{\langle \chi_{W_i}, \chi_{W_i} \rangle}$$

as claimed. \square

Remark 4.30. (a) If G is a finite group and $|G|$ invertible in E , then we will identify the group $R_E(G)$ (Definition 4.6) with the free abelian group on the finite set of irreducible characters of G on E .
(b) If $E \subseteq F$ is an extension of fields and χ a character of a representation V of G over E , then the character of $V \otimes_E F$ is the composition $G \xrightarrow{\chi} E \hookrightarrow F$, and abusing notation we will also write χ for the character of $V \otimes_E F$.

Proposition 4.31. *We continue to assume that $|G|$ is invertible in E . Let $E \subseteq F$ be an extension of fields. A representation of G over F comes from a representation of G over E if and only if its class lies in $R_E(G) \subseteq R_F(G)$ (Proposition 4.11).*

Proof. Clearly, if a representation comes from E , then its class lies in $R_E(G)$. Conversely, let χ be the character of a representation of G over F , and assume that $\chi \in R_E(G)$. Let ψ_1, \dots, ψ_r be the irreducible characters of G over E . By assumption we can write $\chi = \sum_{i=1}^r a_i \psi_i$, with $a_i \in \mathbb{Z}$. Since χ and ψ_i are characters of representations over F (Remark 4.30), we have $\langle \psi_i, \psi_i \rangle > 0$, and

$$a_i \langle \psi_i, \psi_i \rangle = \langle \chi, \psi_i \rangle \geq 0,$$

so $a_i \geq 0$. \square

With the preceding corollary in mind, we make a detour to complex representations.

4.1.4. *Complex representations.* In this section let $E = \mathbb{C}$.

Definition 4.32. Recall that $\mathbf{C}_{\mathbb{C},G}$ denotes the space of class functions $G \rightarrow \mathbb{C}$. For $\varphi, \psi \in \mathbf{C}_{\mathbb{C},G}$ we define

$$(\varphi|\psi)_G = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}.$$

If no confusion is possible we write $(-, -) = (-, -)_G$. This is a scalar product on $\mathbf{C}_{E,G}$: $(\varphi|\psi) = \overline{(\psi|\varphi)}$, $(\varphi|\varphi) > 0$ for $\varphi \neq 0$, and $(\varphi|\psi)$ is linear (resp. semi-linear) in φ (resp. ψ).

Lemma 4.33. *If χ is the character of a complex representation ρ of G , and $\varphi \in \mathbf{C}_{\mathbb{C},G}$, then*

$$(\varphi|\chi) = \langle \varphi, \chi \rangle.$$

Proof. This is clear as $\chi(g)$ is a sum of roots of unity, and if $\xi \in \mathbb{C}$ is a root of unity then $\xi^{-1} = \bar{\xi}$. \square

We have seen that the irreducible characters of G form an orthonormal system in $\mathbf{C}_{\mathbb{C},G}$ with respect to $\langle -, - \rangle_G$. In fact, they also span $\mathbf{C}_{\mathbb{C},G}$.

Theorem 4.34. *Let G be a finite group. Its irreducible characters χ_1, \dots, χ_r form an orthonormal basis of the space of class functions $\mathbf{C}_{\mathbb{C},G}$.*

Proof. Let $\varphi \in \mathbf{C}_{\mathbb{C},G}$ be a class function such that $\langle \varphi, \chi \rangle = 0$ for all irreducible characters χ of G . We want to show that $\varphi = 0$.

If $\rho : G \rightarrow \mathrm{GL}(V)$ is any representation of G , define $\rho_\varphi := \sum_{g \in G} \varphi(g) \rho(g^{-1})$. Then ρ_φ is an endomorphism of V , and it is easily checked that ρ_φ is actually an endomorphism of the representation ρ . If ρ is irreducible, then Schur's Lemma 4.8 shows that $\rho_\varphi = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$. We have

$$\lambda \dim V = \mathrm{Tr}(\rho_\varphi) = |G| \langle \varphi, \chi_\rho \rangle = 0.$$

It follows that $\rho_\varphi = 0$ even if ρ is not irreducible, since ρ is the direct sum of irreducible subrepresentations. Now let ρ be the regular representation of G , i.e. G acts on the $|G|$ -dimensional E -vector space with basis $\{e_g | g \in G\}$ via $\rho(g)(e_h) = e_{gh}$. Then

$$0 = \rho_\varphi(e_1) = \sum_{g \in G} \varphi(g) \rho(g^{-1})(e_1) = \sum_{g \in G} \varphi(g) e_{g^{-1}}.$$

It follows that $\varphi = 0$, which is what we wanted to show. \square

Corollary 4.35. *If G is a finite group, then the number of nonisomorphic irreducible complex representations is equal to the number of conjugacy classes in G . This is also true over any splitting field of characteristic 0.*

Proof. It is clear that the dimension of $\mathbf{C}_{\mathbb{C},G}$ is the number of conjugacy classes in G . By the theorem, this dimension is precisely the number of irreducible characters. If E is a splitting field of characteristic 0, then there exists a splitting field $E' \subseteq E$ which is finitely generated over \mathbb{Q} , such that every representation of G over E comes from E' . The field E' can be embedded in \mathbb{C} , so the claim follows from Proposition 4.16. \square

The following corollary will be crucial for Section 4.2.

Corollary 4.36. *A class function φ is the character of a representation of G if and only if it is a linear combination*

$$\varphi = a_1 \chi_1 + \dots + a_r \chi_r$$

with $a_i \in \mathbb{Z}_{\geq 0}$ and χ_i characters of irreducible representations of G .

4.1.5. Brauer's Theorem. We will need one more tool in our bag before we can return to extensions of local fields. We continue to work over the field of complex numbers.

Definition 4.37. Let $\alpha : H \rightarrow G$ be a morphism of groups. If $\varphi \in \mathbf{C}_{\mathbb{C},G}$, then the composition $\varphi \circ \alpha$ is a class function on H , which we call *the restriction of φ* , and denote by $\alpha^* \varphi$.

Clearly, if φ is the character of a representation ρ of G , i.e. if $\varphi \in R_{\mathbb{C}}^+(G)$, then $\alpha^* \varphi$ is the character of the representation $\mathrm{Res}_\alpha \rho$.

If $\varphi \in \mathbf{C}_{\mathbb{C}, H}$, there is an *induced class function* on G denoted by $\alpha_*\varphi$, which can be computed using the following two special cases:

(a) If α is injective, then

$$\alpha_*\varphi(g) := \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \varphi(xgx^{-1}).$$

(b) If α is surjective, then

$$\alpha_*\varphi(g) := \frac{1}{|\ker(\alpha)|} \sum_{h \mapsto g} \varphi(h).$$

In general we define $\alpha_*\varphi$ by factoring α into a surjective map followed by an injective map.

Lemma 4.38. *If $\alpha : H \rightarrow G$ is a morphism of finite groups and $\rho : H \rightarrow \mathrm{GL}(V)$ a complex representation of H with character χ_ρ , then $\alpha_*\chi_\rho$ is the character of $\mathrm{Ind}_\alpha \rho$.*

Proof. Since α is the composition of an injection and a surjection, it suffices to check that the character of $\mathrm{Ind}_\alpha \rho$ satisfies the two formulas (a) and (b) if α is either injective, or surjective.

In the first case this is a simple calculation, which we leave as an exercise. In the second case, α induces an isomorphism $H/\ker(\alpha) \rightarrow G$. Since $\ker(\alpha)$ is a normal subgroup of H , the subspace $V' := \{v - hv \in V \mid h \in \ker(\alpha), v \in V\}$ is a subrepresentation, and $\ker(\alpha)$ acts trivially on V/V' . It is not difficult to show that the map $\mathrm{Ind}_\alpha \rho = E[H/\ker(\alpha)] \otimes_{E[H]} V \rightarrow V/V'$, defined by $h \ker(\alpha) \otimes v \mapsto hv + V'$ is an isomorphism of H (or $H/\ker(\alpha)$)-representations. Without loss of generality we may assume that ρ is irreducible. Hence $V' = V$ or $V' = 0$. If $V' = 0$, then $\ker(\alpha)$ acts trivially on V , so $\chi_\rho(hh') = \chi_\rho(h)$ if $h' \in \ker(\alpha)$. Hence the formula.

Otherwise, $V' = V$, $\mathrm{Ind}_\alpha \rho = 0$, and we check that $\alpha_*\chi_\rho(g) = 0$. Indeed, if $h_0 \in \ker(\alpha)$ then for all $v \in V'$ we see

$$\sum_{h \in \ker(\alpha)} h(v - h_0v) = 0.$$

As $V = V'$, $\sum_{h \in \ker(\alpha)} hv = 0$ for all $v \in V$. This means that

$$\sum_{h \mapsto g} \mathrm{Tr}_V(h) = \mathrm{Tr}_V \left(\tilde{g} \sum_{h \in \ker(\alpha)} h \right) = 0$$

for every $g \in H/\ker(\alpha)$ with lift $\tilde{g} \in H$. Hence $\alpha_*\chi_\rho = 0$. \square

The adjunction relation between induction and restriction of representations induces a relation for induction and restriction of class functions:

Proposition 4.39 (Frobenius reciprocity). *If $\alpha : H \rightarrow G$ is a homomorphism of groups, ψ a class function on H , φ a class function on G , then*

$$(\psi | \alpha^* \varphi) = (\alpha_* \psi | \varphi).$$

Proof. By Theorem 4.34, we know that ψ is a \mathbb{C} -linear combination of characters of H , and similarly for φ . We may thus assume that $\psi = \chi_W$, $\varphi = \chi_V$. In this case we have

$$\begin{aligned} (\chi_W | \alpha^* \chi_V) &= \dim \operatorname{Hom}_{E[H]}(W, \operatorname{Res}_\alpha V) \\ &= \dim \operatorname{Hom}_{E[G]}(\operatorname{Ind}_\alpha W, V) \\ &= (\alpha_* \chi_W | \chi_V). \end{aligned}$$

□

In practice, a question about a general representation can often be reduced to simpler representations on a different group.

Theorem 4.40 (Brauer). *If G is a finite group and χ a character of a finite dimensional complex representation of G , then χ is a \mathbb{Z} -linear combination of characters of the form $\alpha_{i,*} \chi_i$, where $\alpha_i : H_i \hookrightarrow G$ is the inclusion of a subgroup, and χ_i is the character of a 1-dimensional representation of H .*

Theorem 4.40 follows from a slight variant, also due to Brauer. Recall that if p is a prime number, a group G is called *p-elementary* if G is the direct product of a cyclic group of order prime to p with a p -group.

Theorem 4.41. *Let G be a finite group and $V_p \subseteq R(G)$ the subgroup generated by those characters, which are induced by p -elementary subgroups of G . The index of V_p in $R(G)$ is finite and prime-to- p .*

We do not give a proof of Theorem 4.41 (see [45, Ch. 10, Thm. 18]), but we show how to deduce Theorem 4.40 from this.

Proof of Theorem 4.40. If G is abelian then we know from Proposition 4.5 that every irreducible representation has dimension 1, so we assume that G is not abelian.

By Theorem 4.41, $R(G) = \sum_{p \text{ prime}} V_p$, and thus every character of G is a \mathbb{Z} -linear combination of characters induced by a character of a p -elementary subgroup, for some p . Hence we may assume that G is a p -elementary group. Let $\rho : G \rightarrow \operatorname{GL}(V)$ be an irreducible representation and χ its character. By induction on $|G|$ we may assume that ρ is injective. Write $G = C \times P$, where C is cyclic of order prime to p , and P a p -group. Clearly C is contained in the center of G , and by assumption $G/Z(G)$ is a nonzero p -group. Thus $Z(G/Z(G)) \neq 0$, and $G/Z(G)$ contains a cyclic normal p -group; let $H \subseteq G$ be its preimage in G . Then H is abelian and is not contained in $Z(G)$. Since ρ is injective by assumption, $\rho(H)$ is abelian and not contained in the center of $\rho(G)$, i.e. there exists $h \in H$, such that $\rho(h)$ is not given by multiplication with a scalar. This means that the irreducible decomposition of $\rho|_H : H \rightarrow \operatorname{GL}(V)$ contains nonisomorphic representations, since H is abelian. Let $V = V_1 \oplus \dots \oplus V_n$ be the isotypical decomposition of V as an $E[H]$ -module, i.e. the irreducible representations contained in V_i are all isomorphic, and V_i, V_j contain no common isomorphic irreducible subrepresentations, unless $i = j$. The elements of G permute the V_i transitively, as V is an irreducible $E[G]$ -module.

Define $H' := \{g \in G | gV_1 = V_1\}$. This is a subgroup of G , containing H , and as V is irreducible $H' \neq G$. We claim that ρ is induced by a representation of H' . Indeed, H' acts on V_1 , and $E[G] \otimes_{E[H']} V_1 \xrightarrow{\sim} V$, as $V = \sum_{g \in G} gV_1$.

We have seen that V is induced by a representation of $H' \not\subseteq G$, so the claim follows by induction. \square

Corollary 4.42. *Let G be a finite group of exponent m and E an algebraically closed field of characteristic 0. Let $\zeta_m \in E$ be a primitive m -th root of unity. The injection $R_{\mathbb{Q}(\zeta)}(G) \rightarrow R_E(G)$ is an isomorphism. In other words, $\mathbb{Q}(\zeta)$ is a splitting field for G .*

Proof. Clearly E contains an algebraic closure of \mathbb{Q} , which we can embed into \mathbb{C} . Proposition 4.16 hence reduces the proof to the case $E = \mathbb{C}$. Let χ be the character of a complex representation of G . By Brauer's theorem, χ can be written

$$\chi = \sum_{i=1}^r a_i \alpha_{i,*} \varphi_i$$

with $a_i \in \mathbb{Z}$, $\alpha_i : H_i \hookrightarrow G$ a subgroup and φ_i a 1-dimensional character of H_i . In particular: $\varphi_i : H_i \rightarrow \mathbb{C}^\times$ is a homomorphism, so φ_i has image contained in $\mathbb{Q}(\zeta_m)$ (even in its ring of integers), so φ_i comes from $\mathbb{Q}(\zeta_m)$, and hence so does $\alpha_{i,*} \varphi_i$. \square

Corollary 4.43. *With the notation of the previous corollary, scalar extension induces an equivalence $\text{Rep}_{\mathbb{Q}(\zeta)} G \rightarrow \text{Rep}_E G$.*

Proof. This is just Proposition 4.31 together with Lemma 4.10. \square

4.2. Existence of the complex Artin and Swan representations. We now return to the study of a finite Galois extension L/K of complete discretely valued fields L/K with separable residue extension of degree f . We fix $G := \text{Gal}(L/K)$. Also recall the notations v, v_L for the valuations of K and L , and A, B for the discrete valuation rings of K, L .

Recall that we defined the function $i_G : G \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as $i_G(g) = v_L(gx - x)$. Here $x \in B$ is an element generating B as an A -algebra. Such an element exists by Theorem 3.24. We also saw that $g \in G_n$ if and only if $i_G(g) \geq n+1$, which implies that $i_G(hgh^{-1}) = i_G(g)$, for any $h \in G$, as G_n is normal. Thus i_G is a class function.

Theorem 4.44 (Artin). *In the situation above, the function $a_G : G \rightarrow \mathbb{Z}$,*

$$a_G(g) := \begin{cases} -fi_G(g) & \text{if } g \neq 1 \\ f \sum_{g' \neq 1} i_G(g') & \text{if } g = 1 \end{cases}$$

is the character of a representation of G over the complex numbers.

The proof of this theorem is the object of this section, and it justifies the following definition.

Definition 4.45. The function a_G is called the *Artin character* of G . Once we know that a_G is the character of a representation, we will see in Corollary 4.49 that the function $\text{sw}_G := a_G - (r_G - r_{G/G_0})$ is also a character of G . It is called the *Swan character*. Here we write r_{G/G_0} for the composition $G \rightarrow G/G_0 \xrightarrow{r_{G/G_0}} \mathbb{C}$. If L/K is totally ramified, i.e. if $f = 1$, then $\text{sw}_G = a_G - u_G$, where $u_G = r_G - 1_G$ is the augmentation character of G (see Example 4.23).

Since i_G is a class function, so is a_G , and by Corollary 4.36, Theorem 4.44 is equivalent to the following theorem.

Theorem 4.46. *For any character χ of a complex representation of G , we have $f(\chi) := (a_G|\chi) \in \mathbb{Z}_{\geq 0}$.*

Note that $(a_G|\chi) = (\chi|a_G)$, as a_G takes values in \mathbb{Z} .

The proof of the theorem consists of two steps. First, we show that $(a_G|\chi)$ is always a non-negative *rational* number. Second, to show that $(a_G|\chi)$ is an integer, we use Brauer's theorem 4.40 to reduce to the case where χ is induced by a degree 1 character of a subgroup. If χ is itself a character of a representation ρ of dimension 1, then $G/\ker(\rho)$ is an abelian group corresponding to a Galois subextension K'/K of L/K . In this case, we can relate the number $(a_G|\chi)$ to the jumps in the ramification filtration of $G/\ker(\rho)$, which we know to be integers by the Theorem of Hasse-Arf 3.63.

Let us first see that without loss of generality we may assume that L/K is totally ramified, i.e. that the residue extension is trivial.

Lemma 4.47. *Write $\alpha : G_0 = G^0 \hookrightarrow G$ for the inclusion of the inertia group into G . Then $\alpha_* a_{G_0} = a_G$ and $\text{sw}_G = \alpha_* \text{sw}_{G_0}$.*

Proof. Recall that for $g \in G$, we have

$$\alpha_* a_{G_0}(g) = \frac{1}{|G_0|} \sum_{\substack{x \in G \\ xgx^{-1} \in G_0}} a_{G_0}(xgx^{-1}),$$

according to Definition 4.37. As G_0 is normal in G , $xgx^{-1} \in G_0$ if and only if $g \in G_0$. It follows that for $g \in G \setminus G_0$,

$$\alpha_* a_{G_0}(g) = 0 = -fi_G(g) = a_G(g).$$

Note that if $g \in G_0$, then $i_{G_0}(xgx^{-1}) = i_G(xgx^{-1}) = i_G(g)$. Thus, if $g \neq 1$, we get

$$\alpha_* a_{G_0}(g) = -\frac{|G|}{|G_0|} i_G(g) = a_G(g).$$

Finally, if $g = 1$,

$$\alpha_* a_{G_0}(1) = \frac{|G|}{|G_0|} \sum_{g \in G_0 \setminus \{1\}} i_{G_0}(g) = f \sum_{g \in G \setminus \{1\}} i_G(g) = a_G(1),$$

as $i_G(g) = 0$ for $g \notin G_0$. This proves the statement about a_G .

For sw_G , we just have to note that $\alpha_* r_{G_0} = r_G$ and $\alpha_* 1_{G_0} = r_{G/G_0}$. \square

Hence, for the rest of the section, we assume that L/K is totally ramified, i.e. that $G = G_0$ and $f = 1$.

Lemma 4.48. *For every complex character χ of G , we have $f(\chi) = (a_G|\chi) \in \mathbb{Q}_{\geq 0}$. More precisely, if χ is the character of a representation V , then*

$$f(\chi) = \sum_{i \geq 0} \frac{1}{|G : G_i|} (\dim V - \dim V^{G_i}). \quad (4.3)$$

Proof. First recall that $u_{G_i} = r_{G_i} - 1_{G_i}$ is the augmentation character of the i -th ramification group in the lower numbering. If $\alpha_i : G_i \hookrightarrow G = G_0$ is the inclusion map, then $\alpha_{i,*} u_{G_i}(g) = 0$ if $g \notin G_i$, and if $g \in G_i \setminus \{1\}$ we get

$$\alpha_{i,*} u_{G_i}(g) = -\frac{|G|}{|G_i|} = -|G : G_i|.$$

Hence, if $g \in G_r \setminus G_{r+1}$, then

$$a_G(g) = -(r+1) = \sum_{i=0}^{\infty} \frac{1}{|G : G_i|} \alpha_{i,*} u_{G_i}(g).$$

For $1 \in G$, we compute

$$\sum_{i=0}^{\infty} \frac{\alpha_{i,*} u_{G_i}(1)}{|G : G_i|} = \sum_{i=0}^{\infty} (|G_i| - 1) = \sum_{g \neq 1} i_G(g) = a_G(1),$$

so we have proved that

$$a_G = \sum_{i=0}^{\infty} \frac{1}{|G : G_i|} \alpha_{i,*} u_{G_i}.$$

If φ is a class function on G , then we use Frobenius reciprocity (Proposition 4.39) to compute

$$\begin{aligned} f(\varphi) &= (a_G | \varphi) \\ &= \sum_{i=0}^{\infty} \frac{1}{|G : G_i|} (\alpha_{i,*} u_{G_i} | \varphi) \\ &= \sum_{i=0}^{\infty} \frac{1}{|G : G_i|} (u_{G_i} | \alpha_i^* \varphi) \end{aligned}$$

Recall that $u_{G_i} = r_{G_i} - 1_{G_i}$, and $(r_{G_i} | \alpha_i^* \varphi) = \overline{\varphi(1)}$. We obtain

$$f(\varphi) := \sum_{i=0}^{\infty} \frac{1}{|G : G_i|} (\overline{\varphi(1)} - (1_{G_i} | \alpha_i^* \varphi))$$

Finally, if $\varphi = \chi_V$ for V a $E[G]$ -module, then $\overline{\varphi(1)} = \varphi(1) = \dim V$ and

$$(1_{G_i} | \alpha_i^* \varphi) = \dim_{E[G]}(1_{G_i}, V) = \dim V^{G_i},$$

so we obtain (4.3). \square

Corollary 4.49. *If a_G is the character of a complex representation of G , then so is sw_G .*

Proof. We again assume that L/K is totally ramified, see Lemma 4.47. If χ is the character of the representation V , then $\langle r_G - 1_G, \chi \rangle = \dim V/V^G$. As $G = G_0$, the preceding lemma shows that

$$\langle \text{sw}_G, \chi \rangle = f(\chi) - \dim V/V^G = \sum_{i \geq 1} \frac{1}{|G : G_i|} \dim V/V^{G_i} \geq 0.$$

Thus, if a_G is a character of G , then $f(\chi)$ is an integer and hence $\langle \text{sw}_G, \chi \rangle \in \mathbb{Z}_{\geq 0}$, which implies that sw_G is a character of G . \square

Now we concentrate on characters of 1-dimensional representations.

Lemma 4.50. *Let χ be the character of a 1-dimensional complex representation V of G . Then $\chi(G) \subseteq \mathbb{C}^\times$, and $\chi : G \rightarrow \mathbb{C}^\times$ is a homomorphism of groups. Let $H := \ker(\chi)$ and K'/K the associated subextension of L/K . Write c'_χ for the largest integer for which $(G/H)_{c'_\chi} \neq \{1\}$. Then*

$$f(\chi) = \varphi_{K'/K}(c'_\chi) + 1 \in \mathbb{Z},$$

where $\varphi_{K'/K}$ is the function from Definition 3.44.

Proof. Write $c_\chi := \psi_{L/K'}(c'_\chi)$. From Corollary 3.45 we know that $(G/H)_{c'_\chi}$ is the image of G_{c_χ} in G/H . Then c_χ is the largest integer, such that G_{c_χ} does not act trivially on V . Since V is 1-dimensional, this means that

$$V^{G_i} = \begin{cases} V & \text{if } i > c_\chi \\ 0 & \text{else,} \end{cases}$$

so by (4.3), we obtain

$$f(\chi) = \sum_{i=0}^{c_\chi} \frac{1}{|G : G_i|} = \varphi_{L/K}(c_\chi) + 1,$$

according to (3.6) below Definition 3.44. Finally, we have seen in Proposition 3.51 that

$$\varphi_{L/K}(c_\chi) = \varphi_{K'/K} \varphi_{L/K'}(\psi_{L/K'}(c'_\chi)) = \varphi_{K'/K}(c'_\chi),$$

so we get $f(\chi) = \varphi_{K'/K}(c'_\chi) + 1$ as claimed. Moreover, as $G/H \subseteq \mathbb{C}^\times$, the extension K'/K is abelian, so $\varphi_{K'/K}(c'_\chi) \in \mathbb{Z}$ according to the Hasse-Arf Theorem 3.63. \square

Now we can complete the proof of Theorem 4.46 and hence of Theorem 4.44.

Proof of Theorem 4.46. Let χ be the character of a $\mathbb{C}[G]$ -module V . By Lemma 4.48 we know that $f(\chi) \in \mathbb{Q}_{\geq 0}$. It thus remains to show that $f(\chi)$ is an integer. Brauer's Theorem 4.40 tells us that χ can be written as

$$\chi = \sum_{i=1}^r a_i \chi'_i$$

with $a_i \in \mathbb{Z}$ and $\chi'_i = \gamma_i \chi_i$, where $\gamma_i : H_i \hookrightarrow G$ is a subgroup and χ_i the character of a 1-dimensional representation of H_i . Thus it remains to show that $f(\chi'_i) \in \mathbb{Z}$.

Note that $f(\chi'_i) = (a_G | \chi'_i) = (\gamma_i^* a_G | \chi_i)$ by Frobenius reciprocity, so we have to understand the class function $\gamma_i^* a_G$.

Lemma 4.51. *If $\gamma : H \hookrightarrow G$ is a subgroup, then there exists a non-negative integer λ , such that*

$$\gamma^* a_G = \lambda r_H + a_H.$$

Proof. First let $h \in H \setminus \{1\}$. Then $r_H(h) = 0$, and $i_G(h) = i_H(h)$, so $\gamma^* a_G(h) = \lambda r_H(h) + a_H(h)$ for any λ .

The interesting case is $h = 1$. Recall that A, B denote the valuation rings of K, L . Let x be a generator of B as an A -algebra. Then Theorem 3.23 shows that $\mathfrak{D}_{B/A} = (f'(x))$, if f is the monic minimal polynomial of x . Note that $f'(x) = \prod_{g \neq 1} (x - gx)$, so $v_L(\mathfrak{D}_{B/A}) = a_G(1)$, see Proposition 3.25. Similarly, if $K' = L^H$ and if B' denotes the valuation ring of K' , then $a_H(1) = v_L(\mathfrak{D}_{B/B'})$. Finally, Proposition 3.20 shows that

$$\mathfrak{D}_{B/A} = \mathfrak{D}_{B/B'} \mathfrak{D}_{B'/A},$$

and applying v_L gives

$$a_G(1) = a_H(1) + v_L(\mathfrak{D}_{B'/A}) = a_H(1) + |H| v_{K'}(\mathfrak{D}_{B'/A}).$$

Since $r_H(1) = |H|$, the proof is complete, taking $\lambda = v_{K'}(\mathfrak{D}_{B'/A})$.

□

Returning to the proof of Theorem 4.46, we use the lemma to compute

$$\begin{aligned} f(\chi'_i) &= (\gamma_i^* a_G | \chi_i) \\ &= \lambda(r_{H_i} | \chi_i) + (a_{H_i} | \chi_i) \\ &= \lambda \chi_i(1) + (a_{H_i} | \chi_i) \end{aligned}$$

Since χ_i is the character of a 1-dimensional representation of H , we have $\chi_i(1) = 1$, and we know that $(a_H | \chi_i) \in \mathbb{Z}$, by Lemma 4.50. □

Remark 4.52. Note that the existence of the Artin representation implies the Hasse-Arf theorem: Let L/K be an abelian Galois extension with group G . As before, it suffices to prove Proposition 3.66. We may hence assume that G is cyclic of order n . Fixing a primitive n -th root of unit in E , we get an injective homomorphism $\chi : G \rightarrow E^\times$. Let μ be the largest integer such that $G_\mu \neq 0$. The existence of the Artin representation shows that $f(\chi) = (a_G | \chi) \in \mathbb{Z}$. But as in Lemma 4.50 we see that $f(\chi) = \varphi_{L/K}(\mu) + 1$, so $\varphi_{L/K}(\mu) \in \mathbb{Z}$, which implies Proposition 3.66.

4.3. Integrality over \mathbb{Z}_ℓ . As before, let L/K be a Galois extension of complete discretely valued fields with group G , and assume that the extension of the residue fields is separable of degree f . We have seen in the previous section that the Artin function a_G and the Swan function $\text{sw}_G := a_G - (r_G - r_{G/G_0})$ are characters of complex representations. For this reason, we call these \mathbb{Z} -valued functions the *Artin character* and the *Swan character*. By Proposition 4.19, we know that the attached representations are realizable over $\overline{\mathbb{Q}}$, and hence also over $\overline{\mathbb{Q}_\ell}$, for any prime ℓ . It is therefore meaningful to ask, whether the Artin and Swan representations are defined over a “nice” subfield (or ring) of $\overline{\mathbb{Q}_\ell}$. Answering this question is the main purpose of this section.

Theorem 4.53. *Let ℓ be a prime different from the residue characteristic of K . Then*

- (a) *The Artin and Swan representations are realizable over \mathbb{Q}_ℓ .*
- (b) *There exists a finitely generated projective left- $\mathbb{Z}_\ell[G]$ -module Sw_G , which is unique up to isomorphism, such that $\text{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is isomorphic to the Swan representation, i.e. has character sw_G .*

Remark 4.54. (a) The analogous question about the realizability of sw_G over \mathbb{Q} or even \mathbb{Z} has a negative answer. Indeed, there are even examples of extensions where sw_G is not realizable over \mathbb{R} , [43].
 (b) There is no direct construction of the $\mathbb{Z}_\ell[G]$ -module Sw_G known, but there is a cohomological description of Sw_G , see section 9.2. Note however, that for this construction, one needs (amongst other ingredients), Theorem 4.53.

Clearly (b) of Theorem 4.53 implies (a). In the proof of Theorem 4.53, (b), one needs a good understanding of the representation theory of G over three different bases: \mathbb{Q}_ℓ , \mathbb{Z}_ℓ and \mathbb{F}_ℓ . Until now, we have mainly discussed representations over a field of characteristic prime to the order of the group (recall that even if L/K is totally ramified, G is an extension of a cyclic

group of order prime to p and the p -group G_1 , so $|G|$ will not necessarily be prime to ℓ). A complete presentation of the necessary representation theory is beyond the scope of these notes, but we will nonetheless try to give an overview.

4.3.1. Projective representations. If E is a field and G a group such that $|G|$ is invertible in E , then it is a consequence of Maschke's Theorem 4.3 that two finitely generated $E[G]$ -modules are isomorphic if and only if their classes in $R_E(G)$ agree. This is no longer the case if $|G|$ is not invertible in E (Example 4.4). We can recover this nice property, once we concentrate on *projective* $E[G]$ -modules. We write $P_E(G)$ for the Grothendieck group of projective finitely generated $E[G]$ -modules.

Recall that if R is a (not necessarily commutative) ring, and M a left- R -module, then M is called *projective*, if there exists a free R -module F , such that M is a direct summand of F .

Definition 4.55. Let R be ring.

- A surjective morphism $f : V \rightarrow W$ of left R -modules is an *essential epimorphism* if for every proper submodule $V' \subsetneq V$ we have $f(V') \subsetneq W$.
- An essential epimorphism $f : P \rightarrow V$ with P projective is called *projective cover* or *projective envelope* of V .

Lemma 4.56. Let E be a field, G a finite group and V a finitely generated $E[G]$ -module. Up to unique isomorphism there exists a unique quotient $q_{\max} : V \twoheadrightarrow S_{\max}$ with S_{\max} semi-simple and $\dim_E S_{\max}$ maximal among all semi-simple quotients of V .

If $f : V \twoheadrightarrow S'$ is any quotient with S' semi-simple, then S' is isomorphic to S_{\max} if and only if f is an essential epimorphism.

Proof. Simple quotients of V correspond to maximal submodules of V . Thus, as $\dim_E V < \infty$, there exists a quotient $q_{\max} : V \twoheadrightarrow S_{\max}$ such that S_{\max} is semi-simple and such that its dimension is maximal among the semi-simple quotients of V . If $q : V \twoheadrightarrow S$ is a simple quotient, then $q_{\max} \oplus q : V \rightarrow S_{\max} \oplus S$ is not surjective, and as S is simple, this implies that the image of $q_{\max} \oplus q$ does not contain $0 \oplus S \subseteq S_{\max} \oplus S$, so $\text{im}(q_{\max}) \cap (0 \oplus S) = 0$. It follows that $\ker(q_{\max}) \subseteq \ker(q)$.

Let $\text{rad}(V)$ denote the intersection of all maximal submodules of V . we have proved that $\ker(q_{\max}) \subseteq \text{rad}(V)$. On the other hand, an element $v \in \text{rad}(V)$ maps to 0 in every simple quotient, and hence also in every semi-simple quotient. Thus $\ker(q_{\max}) = \text{rad}(V)$ and up to unique isomorphism, q_{\max} is the quotient morphism $V \twoheadrightarrow V/\text{rad}(V)$.

Note that the maximal semi-simple quotient is an essential epimorphism: Otherwise there would exist a maximal submodule $M \subsetneq V$ such that $q_{\max}(M) = S_{\max}$. This would mean that M maps surjectively onto every simple quotient of V , which is impossible, as $V/M \neq 0$ is simple.

If $q' : V \rightarrow S'$ is a semi-simple quotient, then S' is isomorphic to a direct summand of S_{\max} . In particular $q'(q_{\max}^{-1}(S')) = S'$, so if q' is an essential epimorphism, then $q_{\max}^{-1}(S') = V$ and hence $S_{\max} = S'$. \square

Definition 4.57. Thanks to the lemma, we can talk about the *maximal semi-simple quotient* of V .

Proposition 4.58. *Let G be a group and E a field.*

- (a) *If $|G|$ is invertible in E , then every finitely generated $E[G]$ -module is projective.*
- (b) *For every finitely generated $E[G]$ -module V , there exists a projective envelope $f_V : P_V \rightarrow V$. The pair (P_V, f_V) is unique up to isomorphism.*
- (c) *Every finitely generated projective $E[G]$ -module P is the projective envelope of its maximal semi-simple quotient.*
- (d) *An indecomposable finitely generated projective $E[G]$ -module is the projective envelope of a simple $E[G]$ -module.*
- (e) *If $P_E(G)$ denotes the Grothendieck ring of finitely generated projective $E[G]$ -modules, then $P_E(G)$ is freely generated by the projective envelopes of simple $E[G]$ -modules. In particular, two finitely generated projective $E[G]$ -modules P, P' are isomorphic if and only if their classes in $P_E(G)$ agree.*

Proof. (a) We have seen that if $|G|$ is invertible, then every finitely generated $E[G]$ -module is the direct sum of irreducible $E[G]$ -modules and that an irreducible $E[G]$ -module is a direct summand of the regular representation, which is a free $E[G]$ -module.

- (b) We just give a sketch, for full details see [45, Prop. 41]. If V is a finitely generated $E[G]$ -module, there exists a finitely generated projective $E[G]$ -module Q and a surjection $Q \twoheadrightarrow V$. Let $P \subseteq Q$ be a submodule of minimal length amongst the submodules M for which the composition $M \rightarrow Q \rightarrow V$ is surjective. Clearly the map $P \rightarrow V$ is an essential epimorphism. One proceeds to show that P is a direct summand of Q and hence projective.

The uniqueness of projective envelopes is not difficult to prove.

- (c) This follows directly from the lemma.
- (d) One easily sees that the projective envelope of a direct sum is the direct sum of the envelopes of the summands. We also know from (c) that every finitely generated projective $E[G]$ -module is the projective envelope of a semi-simple $E[G]$ -module. Hence the claim.
- (e) If P is a finitely generated projective $E[G]$ -module, P can be written as a finite direct sum of indecomposable finitely generated projective $E[G]$ -modules, each of which is the projective envelope of a simple $E[G]$ -module V_i . Thus $P_E(G)$ is generated by the projective envelopes of simple $E[G]$ -modules. On the other hand, every extension $0 \rightarrow P_1 \rightarrow P \rightarrow P_2 \rightarrow 0$ of projective $E[G]$ -modules splits.

□

Now let E'/E be an extension of fields; we have seen in Proposition 4.11 that $R_E(G) \rightarrow R_{E'}(G)$ is injective. The same is true for $P_E(G)$:

Proposition 4.59. *The map $P_E(G) \rightarrow P_{E'}(G)$ is injective.*

Proof. We have seen in Proposition 4.11 that $R_E(G) \rightarrow R_{E'}(G)$ is injective. More precisely, we have seen that if V_1, V_2 are nonisomorphic irreducible

$E[G]$ -modules, then $V_1 \otimes_E E'$ and $V_2 \otimes_E E'$ have no common composition factors.

If $f : W \rightarrow V$ is an essential epimorphism of finitely generated $E[G]$ -modules, then so is $f' : W \otimes_E E' \rightarrow V \otimes_E E'$. To see this, fix a basis $\{e_i\}_{i \in I}$ of E' over E . Then $W \otimes_E E'$ considered as a (not necessarily finitely generated) $E[G]$ -module is just $\bigoplus_i W \otimes e_i \cong W^{\oplus |I|}$ and similarly for $V \otimes_E E'$. If $M \subseteq W \otimes_E E'$ is a maximal submodule such that $f'(M) = V \otimes_E E'$, then $M \cap (W \otimes e_i) \twoheadrightarrow V \otimes e_i$. As f is an essential epimorphism it follows that $M = W \otimes_E E'$, and the claim is proved.

Thus, if V_1, V_2 are nonisomorphic irreducible $E[G]$ -modules with projective envelopes P_1, P_2 , then $P_i \otimes E' \rightarrow V_i \otimes E'$, $i = 1, 2$, are projective envelopes. The maximal semi-simple quotient of $P_i \otimes E'$ is the maximal semi-simple quotient of $V_i \otimes E'$ according to Lemma 4.56. It follows that the maximal semi-simple quotients of $P_1 \otimes E'$ and $P_2 \otimes E'$ have no common factors, and thus $P_E(G) \rightarrow P_{E'}(G)$ is injective. \square

4.3.2. Representation theory in mixed characteristic. Now let E be a complete discretely valued field with residue field k of characteristic $\ell > 0$ and G a finite group. If $A := \mathcal{O}_E$ is the valuation ring of E , we write $P_A(G)$ for the Grothendieck ring of finitely generated projective $A[G]$ -modules. Reduction modulo the maximal ideal \mathfrak{m} of A gives us a homomorphism of groups $P_A(G) \rightarrow P_k(G)$.

Proposition 4.60. $P_A(G) \rightarrow P_k(G)$ is an isomorphism.

Proof. We only give a rough sketch of the proof. For details, see [45, §14.4]. If P_1, P_2 are projective $A[G]$ -modules, and $\bar{f} : \bar{P}_1 \rightarrow \bar{P}_2$ is an isomorphism of $k[G]$ -modules, then projectivity implies that \bar{f} lifts to a morphism of $A[G]$ -modules $f : P_1 \rightarrow P_2$, which is an isomorphism by Nakayama's Lemma. This shows that the map $P_A(G) \rightarrow P_k(G)$ is injective.

To see that it is surjective, one uses that A is complete. Let \mathfrak{m} be the maximal ideal of A . Write $A_n := A/\mathfrak{m}^n$. If \bar{P} is a projective $k[G]$ -module, let $P_n \rightarrow \bar{P}$ denote the projective envelope of \bar{P} considered as an $A_n[G]$ -module. It is not difficult to verify that $P_n/\mathfrak{m}P_n \xrightarrow{\cong} \bar{P}$ as $k[G]$ -modules. Passing to the limit over n produces an $A[G]$ -module \hat{P} , which is free as an A -module and whose reduction is isomorphic to the projective $k[G]$ -module \bar{P} . A calculation is required to show that this implies that \hat{P} is actually projective as an $A[G]$ -module. \square

Proposition 4.61. There is a commutative diagram of rings:

$$\begin{array}{ccc} P_A(G) \cong P_k(G) & \xrightarrow{c} & R_k(G) \\ & \searrow e & \nearrow d \\ & R_E(G) & \end{array}$$

compatible with base change along finite extensions $E \subseteq E'$.

Proof. The map c is just given by “forgetting” the projectivity of a module, while e is given by $- \otimes_A E$. Only d requires work: Given a $E[G]$ -module V , pick a finitely generated free A -submodule W of V with $\text{rank}_A W = \dim_E V$.

Such a submodule is called *lattice*. Define $W' := \sum_{g \in G} gW$. This is also a lattice, and an $A[G]$ -module. We claim that the class of $W'/\mathfrak{m}W'$ in $R_k(G)$ is independent of the choice of the lattice W (the isomorphism class of the representation $W'/\mathfrak{m}W'$ is not necessarily independent of the choice of W !). Let W_1, W_2 be G -stable lattices in V , without loss of generality we may assume $W_2 \subseteq W_1$: There exists some $r \geq 0$, such that $\mathfrak{m}^r W_2 \subseteq W_1$.

There exists a minimal n such that $\mathfrak{m}^n W_1 \subseteq W_2$. We proceed by induction on n . If $n = 0$, there is nothing to show. If $n = 1$, the module W_1/W_2 is killed by multiplication with a uniformizer π of A . This means we have an short exact sequence of $k[G]$ -modules

$$0 \rightarrow W_1/W_2 \xrightarrow{\pi} W_2/\mathfrak{m}W_2 \rightarrow W_1/\mathfrak{m}W_1 \rightarrow W_1/W_2 \rightarrow 0$$

This means that the classes of $W_1/\mathfrak{m}W_1$ and $W_2/\mathfrak{m}W_2$ agree in $R_k(G)$, as claimed.

If $n > 1$, take $W_3 := \mathfrak{m}^{n-1}W_1 + W_2$. Then $\mathfrak{m}W_3 \subseteq W_2 \subseteq W_3$, so by the previous calculation the classes of $W_2/\mathfrak{m}W_2$ and $W_3/\mathfrak{m}W_3$ agree in $R_k(G)$. On the other hand, $\mathfrak{m}^{n-1}W_1 \subseteq W_3 \subseteq W_1$, so by induction the classes of $W_3/\mathfrak{m}W_3$ and $W_1/\mathfrak{m}W_1$ agree. This concludes the construction of the map $d: R_E(G) \rightarrow R_k(G)$.

It is easy to see that this map is a ring homomorphism, and the compatibility with base change is also clear. \square

Exercise 4.62. The triangle (4.61) is not so interesting if the order $|G|$ is prime to the residue characteristic $\text{char}(k) = \ell$. Indeed, under that assumption, prove that

- (a) Every finitely generated $k[G]$ -module is projective.
- (b) Every finitely generated $A[G]$ -module which is free over A is projective.
- (c) A projective $A[G]$ -module V is simple if and only if $V \otimes_A k$ is a simple $k[G]$ -module.
- (d) The maps c, d, e in the triangle (4.61) are isomorphisms.

We will give – without proof – some important properties of the maps (4.61).

Theorem 4.63.

- (a) The map $d: R_E(G) \rightarrow R_k(G)$ is surjective.
- (b) The map $e: P_k(G) \rightarrow R_E(G)$ is injective.
- (c) Let E' be a finite extension of the complete, discretely valued field E , and write k' for its residue field. We then have a commutative diagram

$$\begin{array}{ccc} P_{k'}(G) & \xrightarrow{e'} & R_{E'}(G) \\ \uparrow & & \uparrow \\ P_k(G) & \xrightarrow{e} & R_E(G), \end{array}$$

and a class $x \in R_{E'}(G)$ lies in $\text{im}(e)$ if and only if its character χ_x has values in E and if $\chi_x(g) = 0$ whenever $\ell \mid \text{ord}(g)$.

Here, the character χ_x is to be understood as follows: E' has characteristic 0, and therefore x can be uniquely written a \mathbb{Z} -linear combination of classes $[V_i]$ of irreducible E' -representations; $x = \sum_{i=1}^n a_i [V_i]$, $a_i \in \mathbb{Z}$. We then define $\chi_x := \sum_{i=1}^n a_i \chi_{V_i} : G \rightarrow E'$.

The proof of Theorem 4.63 is long and rather complicated, see [45, Ch. 17]. We omit it. Note that statement (c) can be understood as a generalization of Proposition 4.31.

Admitting the theorem, we can prove Proposition 4.64, which was the reason for our discussion.

Proposition 4.64. *We continue to denote by E a complete discretely valued field of characteristic 0 with valuation ring A and residue field k of characteristic $\ell > 0$*

Let E' be a finite extension of E and A' the ring of integers of E' . A class $x \in R_{E'}(G)$ arises from a projective $A[G]$ -module if and only if

- (a) *The character χ_x takes values in E .*
- (b) *There exists $n \in \mathbb{N}$, such that nx is the class of a E' -representation, which comes from a projective $A'[G]$ -module.*

Moreover, the projective $A[G]$ -module realizing x is unique up to isomorphism.

Proof. Clearly, if the class x comes from a projective $A[G]$ -module, then (a) and (b) are satisfied.

Conversely, assume that $x \in R_{E'}(G)$ satisfies (a) and (b). Let V' be the projective $A'[G]$ -module corresponding to nx . Write V for the $A[G]$ -module obtained from V' via the inclusion $A \subseteq A'$. Then V is a projective $A[G]$ -module as A' is free over A .

Since E'/E is finite, $V \otimes_A E$ is an E -vector space of dimension $[E' : E] \cdot \dim_{E'}(V' \otimes_{A'} E')$. We compute the character of $V \otimes_A E$. Let $b_1, \dots, b_r \in E'$ be a basis of E' as a E -vector space such that $b_1 = 1$, and fix $g \in G$. If $v' \in V'$ is an eigenvector of g acting on V' , then $v, b_2 v, b_3 v, \dots$ are linearly independent eigenvectors of g acting on V . Thus $\text{Tr}(g|V \otimes_A E) = [E' : E] \text{Tr}(g|V' \otimes_{A'} E')$. Using the notation from Proposition 4.61, we see that $e([V]) = [E' : E]nx \in R_{E'}(G)$. But by construction $e([V]) \in R_E(G) \subseteq R_{E'}(G)$, so by Theorem 4.63 we see that $\chi_{e([V])}(g) = 0$ whenever $\ell \nmid \text{ord}(g)$. This means that $\chi_x(g) = 0$ whenever $\ell \nmid \text{ord}(g)$. Again by Theorem 4.63, it follows that $x \in \text{im}(P_A(G) \xrightarrow{e} R_E(G))$. Let $y \in P_A(G)$ be an element such that $e(y) = x$. Since e is injective, $[V] = [E' : E]ny$. As $V \in P_A^+(G)$, the same is true for y , i.e. there exists a projective $A[G]$ -module Y , such that $Y \otimes_A E' \cong V'$, i.e. $[Y \otimes_A E] = x \in R_E(G)$.

The projective $A[G]$ -module Y is unique up to isomorphism by Proposition 4.58, (e): The isomorphism class of Y is determined by its class in $P_A(G)$, and by the injectivity of e , this class is uniquely determined. \square

4.3.3. *The Swan character comes from \mathbb{Z}_ℓ .* Using Proposition 4.64, we can now finally prove the ℓ -adic integrality of the Swan conductor.

Proof of Theorem 4.53. Recall that we defined $\text{sw}_G := a_G - (r_G - r_{G/G_0})$, and that we saw that a_G , hence sw_G , can be realized over a finite extension E/\mathbb{Q}_ℓ . We want to apply Proposition 4.64. We know that sw_G takes values

in \mathbb{Z} , so to prove that sw_G can be realized by a projective $\mathbb{Z}_\ell[G]$ -module, we just have to check that some multiple $n\text{sw}_G$ can be realized by a projective $\mathcal{O}_E[G]$ -module, where \mathcal{O}_E is the integral closure of \mathbb{Z}_ℓ in E . To see this, note that by Lemma 4.47 we know that sw_G is induced by sw_{G_0} , so we may assume that L/K is totally ramified, i.e. that $G_0 = G$. We computed in Lemma 4.48 that

$$a_G = \sum_{i=0}^{\infty} \frac{1}{|G : G_i|} \alpha_{i,*} u_{G_i},$$

where $\alpha_i : G_i \hookrightarrow G$ are the inclusions of the ramification subgroups. It follows that

$$|G|\text{sw}_G = \sum_{i=1}^{\infty} |G_i| \alpha_{i,*} u_{G_i}.$$

Since $u_{G_i} = r_{G_i} - 1$, each $\alpha_{i,*} u_{G_i}$ can be realized over $\mathcal{O}_E[G]$ (even over $\mathbb{Z}_\ell[G]$). It remains to see that the augmentation representations U_{G_i} , $i > 0$, are projective $\mathbb{Z}_\ell[G_i]$ -modules.

As the groups G_i are p -groups, for $i > 0$, and since $p \neq \ell$, every finitely generated $\mathbb{Z}_\ell[G_i]$ -module V which is free (=torsion-free) over \mathbb{Z}_ℓ is projective. Indeed, one finds a surjection $F \twoheadrightarrow V$, with F a free $\mathbb{Z}_\ell[G_i]$ -module, and as $(|G_i|, \ell) = 1$ for $i > 0$, this surjection splits, and V is a direct summand of F . In particular the augmentation representation of G_i is projective over $\mathbb{Z}_\ell[G_i]$ for $i > 0$.

It follows that $|G|\text{sw}_G$ can be realized by a projective $\mathcal{O}_E[G]$ -module, and Proposition 4.64 shows that sw_G can be realized by a finitely generated projective $\mathbb{Z}_\ell[G]$ -module, unique up to isomorphism. \square

4.4. Swan conductor of an ℓ -adic Galois representation. In this section let K be a complete discretely valued field with perfect residue field k of characteristic $p > 0$. Let K^{sep} be a separable closure of K , and write $G_K := \text{Gal}(K^{\text{sep}}/K)$. Fix a second prime number $\ell \neq p$. We are now interested in *continuous representations* $\rho : G_K \rightarrow \text{GL}(V)$, where V is a finite dimensional vector space over a finite extension E of \mathbb{Q}_ℓ , equipped with its ℓ -adic topology. Such a representation is called *ℓ -adic Galois representation*.

Definition 4.65. Let us recall the definition of the topology on $\text{GL}(V)$: The topology on V makes $\text{End}(V)$ into a topological ring, and we can identify the group $\text{GL}(V)$ with the subspace of $\text{End}(V) \times \text{End}(V)$ defined by $XY = 1$, i.e. with the set of pairs (A, A^{-1}) , $A \in \text{GL}(V)$. Using this identification we equip $\text{GL}(V)$ with the subspace topology to make it a topological group. Analogously, if \mathcal{V} is a free \mathcal{O}_E -module of finite rank, we can make $\text{Aut}_{\mathcal{O}_E}(\mathcal{V})$ into a topological group.

Lemma 4.66. *Let E be a finite extension of \mathbb{Q}_ℓ , V a finite dimensional E -vector space, G a profinite group and $\rho : G \rightarrow \text{GL}(V)$ continuous representation. Writing \mathcal{O}_E for the ring of integers of E , there exists a free \mathcal{O}_E -submodule $\mathcal{V} \subseteq V$, such that $V = \mathcal{V} \otimes_{\mathcal{O}_E} E$ and such that ρ factors*

$$\rho : G \rightarrow \text{GL}(\mathcal{V}) \rightarrow \text{GL}(V),$$

where $\text{GL}(\mathcal{V}) := \text{Aut}_{\mathcal{O}_E}(\mathcal{V})$.

Proof. Let e_1, \dots, e_r be a basis of V and use it to identify $\mathrm{GL}(V)$ with $\mathrm{GL}_r(E)$. The inclusion $\mathrm{GL}_r(\mathcal{O}_E) \subseteq \mathrm{GL}_r(E)$ given by the choice of the basis makes $\mathrm{GL}_r(\mathcal{O}_E)$ into an *open* subgroup of $\mathrm{GL}_r(E)$. Then $\{\rho^{-1}(M \mathrm{GL}_r(\mathcal{O}_E)) | M \in \mathrm{GL}_r(E)\}$ is an open covering of G , and as G is compact, it follows that there exists a minimal $n \geq 1$ and matrices $M_1, \dots, M_n \in \mathrm{GL}_r(E)$, such that $\mathrm{im}(\rho) \subseteq M_1 \mathrm{GL}_r(\mathcal{O}_E) \cup \dots \cup M_n \mathrm{GL}_r(\mathcal{O}_E)$.

If $\mathcal{V}' := \sum_{i=1}^r e_i \mathcal{O}_E \subseteq V$, then for every $g \in G$ there exists i such that $\rho(g)(\mathcal{V}') = M_i \mathcal{V}'$, and conversely, for every i there exists $g \in G$ such that $M_i \mathcal{V}' = \rho(g) \mathcal{V}'$.

Defining $\mathcal{V} := \sum_{i=1}^n M_i \mathcal{V}' \subseteq V$ hence gives a G -stable, free sub- \mathcal{O}_E -module of V , such that $\mathcal{V} \otimes_{\mathcal{O}_E} E = V$. \square

We will be particularly interested in how the *wild ramification subgroup* acts in an ℓ -adic representation. Recall that the upper numbering filtration on finite Galois groups is compatible with taking quotients (Herbrand's Theorem, Proposition 3.53), which allowed us to introduce the *upper numbering filtration* on the infinite, absolute Galois group G_K (Definition 3.54). We saw that $G_K^0 \trianglelefteq G_K$ is the kernel of the projection $G_K \twoheadrightarrow \mathrm{Gal}(\bar{k}/k)$. If L/K is a finite Galois extension, we also showed that $\mathrm{Gal}(L/K)_1 = \mathrm{Gal}(L/K)^{\varphi_{L/K}(1)}$ is the unique p -Sylow subgroup of $\mathrm{Gal}(L/K)$, and that the quotient $\mathrm{Gal}(L/K)/\mathrm{Gal}(L/K)_1$ is cyclic of order prime-to- p (Corollary 3.38). Passing to the inverse limit over all finite Galois extensions we obtain a closed normal subgroup $P_K \trianglelefteq G_K$, such that P_K is a pro- p -group, and such that G_K/P_K is pro-cyclic with every finite quotient of order prime to p . In particular, if $H \trianglelefteq G_K$ is an open (hence finite index) normal subgroup, then the image of P_K in the finite Galois group G_K/H is $(G_K/H)_1$.

Definition 4.67. The pro- p -group P_K is called the *wild inertia group* or *wild ramification group*. Let R be a commutative ring and $\rho : G_K \rightarrow \mathrm{GL}_r(R)$ a homomorphism.

- (a) ρ is called *unramified* if $G_K^0 \subseteq \ker(\rho)$.
- (b) ρ is called *tame* or *tamely ramified* if $P_K \subseteq \ker(\rho)$. Otherwise ρ is called *wild* or *wildly ramified*.

Our main interest will be in the cases $R = \mathcal{O}_E, E$ or \mathbb{F}_λ , and ρ continuous.

We now reduce a Galois representation modulo ℓ .

Definition 4.68. Let E be a finite extension of \mathbb{Q}_ℓ with valuation ring \mathcal{O}_E , uniformizer λ , and finite residue field \mathbb{F}_λ . If \mathcal{V} is a free finitely generated \mathcal{O}_E -module, and $\rho : G_K \rightarrow \mathrm{GL}(\mathcal{V})$ a continuous representation, then the composition $\bar{\rho} : G_K \rightarrow \mathrm{GL}(\mathcal{V}) \rightarrow \mathrm{GL}(\bar{\mathcal{V}})$, with $\bar{\mathcal{V}} = \mathcal{V}/\lambda\mathcal{V}$, is called the *reduction modulo λ* of ρ .

The next lemma shows that in the situation of the definition, the pro- p -group $P_K \subseteq G_K$ acts on \mathcal{V} and $\bar{\mathcal{V}}$ through the same *finite* group.

Lemma 4.69. Let $\ell \neq p$ be two primes, E a finite extension of \mathbb{Q}_ℓ , \mathcal{O}_E its valuation ring and \mathbb{F}_λ its residue field. If P is a pro- p -group and $\rho : P \rightarrow \mathrm{GL}_r(\mathcal{O}_E)$ a continuous representation, then the image of ρ is finite, and $\rho(P) \cap \ker(\mathrm{GL}_r(\mathcal{O}_E) \twoheadrightarrow \mathrm{GL}_r(\mathbb{F}_\lambda)) = \{1\}$.

If $\rho : P \rightarrow \mathrm{GL}_r(E)$ is a continuous representation, then ρ factors through a finite quotient of P .

Proof. In Lemma 4.66, we saw that any continuous map $P \rightarrow \mathrm{GL}_r(E)$ factors through (a conjugate of) the inclusion $\mathrm{GL}_r(\mathcal{O}_E) \subseteq \mathrm{GL}_r(E)$, so the final statement follows from the first.

The kernel of the projection $\mathrm{GL}_r(\mathcal{O}_E) \rightarrow \mathrm{GL}_r(\mathbb{F}_\lambda)$ is an open subgroup which can be written as $H := \mathrm{id} + \lambda M_r(\mathcal{O}_E)$, where λ is a uniformizer of \mathcal{O}_E and $M_r(\mathcal{O}_E)$ the ring of $r \times r$ -matrices with entries in \mathcal{O}_E . By continuity $\rho^{-1}(H)$ is an open subgroup of P containing the kernel of ρ . The group H is a pro- ℓ -group, and it is not difficult to see that there are no non-trivial maps between pro- p and pro- ℓ -groups, if $\ell \neq p$. It follows that $\rho^{-1}(H) \subseteq \ker(\rho)$, and hence $\rho(P) \cap H = \{1\}$. As H has finite index in $\mathrm{GL}_r(\mathcal{O}_E)$, $\ker(\rho)$ has finite index in P , which completes the proof. \square

Corollary 4.70. *If \mathcal{V} is a free \mathcal{O}_E -module and $\rho : G_K \rightarrow \mathrm{GL}(\mathcal{V})$ a continuous representation, then ρ is tame if and only if the reduction $\bar{\rho} : G_K \rightarrow \mathrm{GL}(\mathcal{V}) \rightarrow \mathrm{GL}(\bar{\mathcal{V}})$ is tame.*

Proof. The lemma shows that $\rho(P_K) \cong \bar{\rho}(P_K)$. \square

We now use the Swan representation to define a measure for “how wild” a representation is.

Definition 4.71. Let E be a finite extension of \mathbb{Q}_ℓ with valuation ring \mathcal{O}_E , uniformizer λ , and residue field \mathbb{F}_λ . Let \mathcal{V} be a free finitely generated \mathcal{O}_E -module, and $\rho : G_K \rightarrow \mathrm{GL}(\mathcal{V})$ a continuous representation.

- (a) Since \mathbb{F}_λ is a finite field, $G := G_K / \ker(\bar{\rho})$ is the Galois group of a finite extension L/K . Following Serre, we define

$$b(\rho) := b(\bar{\rho}) := \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \bar{\rho}).$$

We also write $b(\mathcal{V})$, if the representation ρ is implicitly given. Note that since Sw_G is a projective $\mathbb{Z}_\ell[G]$ -module, $\overline{\mathrm{Sw}_G}$ is a projective $\mathbb{F}_\lambda[G]$ -module, so the number $b(\mathcal{V})$ only depends on the class of $\bar{\mathcal{V}}$ in $R_{\mathbb{F}_\lambda}(G)$.

- (b) Let V be a finite dimensional E -vector space and $\rho : G_K \rightarrow \mathrm{GL}(V)$ a continuous representation. If $\mathcal{V} \subseteq V$ is a G_K -stable \mathcal{O}_E -lattice for V , then the number $b(\mathcal{V})$ only depends on $\rho : G_K \rightarrow \mathrm{GL}(V)$, not on the choice of the lattice. Indeed, since Sw_G is projective, $b(\mathcal{V})$ only depends on the class of $\bar{\mathcal{V}}$ in $R_{\mathbb{F}_\lambda}(G)$, which only depends on ρ (same argument as in proof as Proposition 4.61). We denote this number by $b(V)$ or $b(\rho)$.

Remark 4.72.

- (a) In the definition of $b(\rho)$ we used the group $G = G_K / \ker(\bar{\rho})$. Instead, we could have used G_K / N for any open normal subgroup N of G_K contained in $\ker(\bar{\rho})$ without changing the result. See the final comments in the proof of Theorem 4.85.
- (b) Taking this into account, we see that in the situation of Definition 4.71, if ρ itself factors through a finite quotient G of G_K , then

$$\begin{aligned} b(\rho) &= \dim_{\mathbb{F}_\lambda[G]} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \bar{\rho}) \\ &= \mathrm{rank}_{\mathcal{O}_E} \mathrm{Hom}_{\mathcal{O}_E[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathcal{O}_E, \rho) \\ &= \dim_E \mathrm{Hom}_{E[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Q}_\ell} E, \rho \otimes E). \end{aligned}$$

To see this, apply the exact functor $\mathrm{Hom}_{\mathcal{O}_E[G]}(\mathrm{Sw}_G \otimes_{\mathbb{Z}_\ell} \mathcal{O}_E, -)$ to the short exact sequence of $\mathcal{O}_E[G]$ -modules

$$0 \rightarrow \mathcal{V} \xrightarrow{\cdot \lambda} \mathcal{V} \rightarrow \overline{\mathcal{V}} \rightarrow 0.$$

Proposition 4.73. *In the situation of the definition, let \mathcal{V} be a finitely generated, free \mathcal{O}_E -module equipped with a continuous representation $\rho : G_K \rightarrow \mathrm{GL}(\mathcal{V})$. If $G := G_K / \ker(\bar{\rho})$, then*

$$b(\mathcal{V}) = \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}} / \overline{\mathcal{V}}^{G_i}) \quad (4.4)$$

where $\overline{\mathcal{V}} = \mathcal{V} / \lambda \mathcal{V}$, with λ a uniformizer of \mathcal{O}_E .

Proof. We have seen that $|G_0| \cdot \mathrm{sw}_G = \sum_{i \geq 1} |G_i| u_{G_i}^*$, where $u_{G_i}^*$ denotes the character of the augmentation representation (Example 4.1, (c)) U_{G_i, \mathcal{O}_E} of G_i , induced along the inclusion $G_i \subseteq G$. It follows that

$$\begin{aligned} |G_0| b(\mathcal{V}) &= \dim_{\mathbb{F}_\lambda}(\mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\overline{\mathrm{Sw}_G^{\oplus |G_0|}}, \overline{\mathcal{V}})) \\ &= \sum_{i \geq 1} \dim_{\mathbb{F}_\lambda}(\mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\overline{\mathrm{Ind}_{G_i}^G U_{G_i, \mathcal{O}_E}^{\oplus |G_i|}}, \overline{\mathcal{V}})) \\ &= \sum_{i \geq 1} |G_i| \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\overline{\mathrm{Ind}_{G_i}^G U_{G_i, \mathcal{O}_E}}, \overline{\mathcal{V}}) \end{aligned}$$

As the G_i , $i > 0$, are p -groups, the augmentation representation is a projective $\mathcal{O}_E[G_i]$ -module (Exercise 4.62), its reduction is the augmentation representation of G_i over \mathbb{F}_λ , because we already saw that $P_A(G_i) \rightarrow P_k(G_i)$ is an isomorphism. It is easily checked that induction and reduction modulo λ commute, so $\overline{\mathrm{Ind}_{G_i}^G U_{G_i, \mathcal{O}_E}} = \mathrm{Ind}_{G_i}^G U_{G_i, \mathbb{F}_\lambda}$.

Finally, note that by the properties of induction

$$\mathrm{Hom}_{\mathbb{F}_\lambda[G]}(\mathrm{Ind}_{G_i}^G U_{G_i, \mathbb{F}_\lambda}, \overline{\mathcal{V}}) = \mathrm{Hom}_{\mathbb{F}_\lambda[G_i]}(U_{G_i, \mathbb{F}_\lambda}, \mathrm{Res}_{G_i}^G \overline{\mathcal{V}}) = \overline{\mathcal{V}} / \overline{\mathcal{V}}^{G_i}$$

The claim follows. \square

By definition, a representation \mathcal{V} of G_K is tame, if P_K acts trivially. On the other hand, the invariant $b(\mathcal{V})$ only depends on the reduction of \mathcal{V} modulo the maximal ideal. Nonetheless, as promised, $b(\mathcal{V})$ does measure whether \mathcal{V} is tame or not.

Proposition 4.74. *Let $\rho : G_K \rightarrow \mathrm{GL}_r(\mathcal{O}_E)$ be a continuous representation. Then the following are equivalent:*

- (a) $\rho \otimes E : G_K \xrightarrow{\rho} \mathrm{GL}_r(\mathcal{O}_E) \hookrightarrow \mathrm{GL}_r(E)$ is tame.
- (b) ρ is tame.
- (c) $\bar{\rho} : G_K \xrightarrow{\rho} \mathrm{GL}_r(\mathcal{O}_E) \twoheadrightarrow \mathrm{GL}_r(\mathbb{F}_\lambda)$ is tame.
- (d) $b(\rho) = 0$.

Proof. The equivalence of (a) and (b) is clear, while the equivalence of (b) and (c) is Corollary 4.70. The equivalence of (c) and (d) follows from the formula of Proposition 4.73. \square

4.4.1. *The break decomposition.* We continue to use the notations from the previous section. The simple observation made in Lemma 4.69 tells us that in any ℓ -adic representation $\rho : G_K \rightarrow \mathrm{GL}_r(E)$, where E is a finite extension of \mathbb{Q}_ℓ , $\ell \neq p$, the wild inertia group P_K always acts through a finite quotient.

We follow Katz [25, Ch. 1] in studying fairly general representations of the pro- p -group P_K (Definition 4.67), which factor through a finite quotient. See also [32, §2].

Let us first establish some more facts about the ramification filtration on G_K .

Lemma 4.75. *For $\lambda \in \mathbb{R}_{\geq 0}$, write*

$$G_K^{\lambda+} := \overline{\bigcup_{\lambda' > \lambda} G_K^{\lambda'}}$$

where $\overline{(-)}$ denotes the closure in the topological group G_K .

The decreasing filtration G_K^λ , $\lambda \in \mathbb{R}_{\geq 0}$, satisfies the following properties.

- (a) $\bigcap_{\lambda > 0} G_K^\lambda = \{1\}$
- (b) For $\lambda > 0$ we have

$$G_K^\lambda = \bigcap_{0 < \lambda' < \lambda} G_K^{\lambda'}.$$

- (c) $P_K = G_K^{0+}$

Proof. (a) If L/K is a finite Galois extension, the quotient map $G_K \twoheadrightarrow \mathrm{Gal}(L/K)$ induces surjections $G_K^\lambda \rightarrow \mathrm{Gal}(L/K)^\lambda$. Thus, if $g \in G_K$ lies in G_K^λ for all $\lambda > 0$, then g maps to 1 in $\mathrm{Gal}(L/K)$ for every finite Galois extension L/K . The claim follows.

- (b) The profinite group G_K/G_K^λ corresponds to an algebraic extension L/K (not necessarily finite) and the ramification filtration on its Galois group G_K/G_K^λ is given by the image of the filtration of G_K . Thus the claim follows from (a).

- (c) Let L/K be a finite Galois extension. If $\varepsilon > 0$, then $\psi_{L/K}(\varepsilon) > 0$, so $\mathrm{Gal}(L/K)^\varepsilon = \mathrm{Gal}(L/K)_{[\psi_{L/K}(\varepsilon)]} \subseteq \mathrm{Gal}(L/K)_1$. Since the image of P_K in $\mathrm{Gal}(L/K)$ is precisely $\mathrm{Gal}(L/K)_1$ it follows that $G_K^{0+} \subseteq P_K$. But we see more: there exists an $\varepsilon_0 > 0$, such that $\mathrm{Gal}(L/K)^{\varepsilon_0} = \mathrm{Gal}(L/K)_1$, so the closed subgroups $G_K^{0+} \subseteq P_K$ have the same images in every finite quotient, which shows that they are equal. (Recall that a closed subgroup H of a profinite group $\varprojlim_N G/N$ is also profinite and isomorphic to $\varprojlim_N H/(N \cap H)$)

□

Definition 4.76. For convenience, we say that a P_K -module is a $\mathbb{Z}[1/p]$ -module M , together with a morphism $\rho : P_K \rightarrow \mathrm{Aut}_{\mathbb{Z}}(M)$, which factors through a finite discrete quotient. A morphism of P_K -modules is a morphism of $\mathbb{Z}[1/p]$ -modules compatible with the action of P_K .

Proposition 4.77 ([25, Ch. I, Prop. 1.1]). *For readability, write $G := G_K$ and $P := P_K$. Let M be a P -module.*

- (a) *There exists a unique decomposition $M = \bigoplus_{x \in \mathbb{R}_{\geq 0}} M(x)$ of P -modules, such that*

- (i) $M(0) = M^P$
- (ii) $M(x)^{G^x} = 0$ for $x > 0$.
- (iii) $M(x)^{G^y} = M(x)$ for $y > x$.
- (b) $M(x) = 0$ for all but finitely many $x \in \mathbb{R}_{\geq 0}$.
- (c) For every $x \in \mathbb{R}_{\geq 0}$, the assignment $M \mapsto M(x)$ is an exact endofunctor on the category of P -modules.
- (d) $\text{Hom}_P(M(x), M(y)) = 0$ unless $x = y$.

Proof. Let $\rho : P \rightarrow \text{Aut}_{\mathbb{Z}}(M)$ be the representation implicit in saying that M is a P -module and let $H = \text{im}(\rho)$. By assumption, H is a finite discrete group. For $x \in \mathbb{R}_{\geq 0}$, let $H(x) := \rho(G_K^x)$, and $H(x+) := \rho(G_K^{x+}) = \bigcup_{y>x} H(y)$. In particular, $H = H(0+)$. Note that for every $x \in \mathbb{R}_{\geq 0}$, $H(x)$ and $H(x+)$ are normal subgroups of H .

For every $x \in \mathbb{R}_{\geq 0}$, we define elements of $\mathbb{Z}[1/p][H]$

$$\pi(x) := \frac{1}{|H(x)|} \sum_{h \in H(x)} h \quad \text{and} \quad \pi(x+) := \frac{1}{|H(x+)|} \sum_{h \in H(x+)} h$$

We have seen such elements before: They define splittings of the projections $\mathbb{Z}[1/p][H(x)] \twoheadrightarrow \mathbb{Z}[1/p]$, $h \mapsto 1$, and similarly for $H(x+)$.

Lemma 4.78. *Almost all the elements $\pi(x+)(1 - \pi(x))$ are 0, and the nonzero elements of the set $\{\pi(x+)(1 - \pi(x)) | x > 0\} \cup \{\pi(0+)\}$ form a set of orthogonal, central idempotents in $\mathbb{Z}[1/p][H]$, whose sum is 1.*

Proof. An easy calculation shows that for all $x \geq 0$, $\pi(x)$ and $\pi(x+)$ are idempotents. They are central as the groups $H(x)$, $H(x+)$ are normal subgroups of H . It follows that the same is true for $\pi(x+)(1 - \pi(x))$ and $\pi(0+)$. For the orthogonality, note that $\pi(x+)\pi(x) = \pi(x)$, so $\pi(x+)(1 - \pi(x)) = \pi(x+) - \pi(x)$. For $y > x$ we compute

$$\begin{aligned} (\pi(x+) - \pi(x))(\pi(y+) - \pi(y)) &= \pi(x+)\pi(y+) - \pi(x+)\pi(y) \\ &= \pi(x+) - \pi(x+)\pi(y), \end{aligned}$$

which is 0 if $\pi(y) \neq \pi(x)$. If $\pi(x) = \pi(y)$, then $H(x) = H(y)$, so $H(x) = H(x+)$ and hence $\pi(x+) - \pi(x) = 0$.

Since H is a finite group, we have $H(x) = H(x+)$ for almost all $x \in \mathbb{R}_{\geq 0}$, thus only finitely many $\pi(x+)(1 - \pi(x))$ are nonzero.

Finally, let us compute the sum of the idempotents:

$$\sum_{x>0} \pi(x+)(1 - \pi(x)) + \pi(0+) = \sum_{n=1}^r (\pi(x_n+) - \pi(x_n)) + \pi(0+) \quad (4.5)$$

where $x_1 < x_2 < \dots < x_r$ are the finitely many elements of $\mathbb{R}_{>0}$ where $H(x_n+) \neq H(x_n)$. Note that in this case

$$H(0+) = H(x_1), \quad H(x_{n-1}+) = H(x_n), n \in [2, r], \quad H(x_r+) = \{1\}.$$

It follows that the coefficient of $h \in H(x_n) \setminus H(x_n+)$ in (4.5) is

$$\underbrace{-\frac{1}{|H(x_n)|} + \frac{1}{|H(x_{n-1}+)|}}_{=0} - \underbrace{\frac{1}{|H(x_{n-1})|} + \frac{1}{|H(x_{n-2}+)|}}_{=0} + \dots = 0$$

A similar computation shows that the coefficient of 1 in (4.5) is 1, so the proof is complete. \square

From the lemma we obtain the decomposition of M that we were looking for: For $x > 0$, $M(x) := \{m \in M \mid \pi(x_+)(1 - \pi(x))m = m\}$, and $M(0) := \{m \in M \mid \pi(0+)m = m\}$. By construction, this is a decomposition into finitely many subrepresentations.

We verify that it has the desired properties. For property (i), we just have to observe that $H(0+) = \text{im}(P)$, as $P = G_K^{0+}$. For (iii), let x_n be defined as in the previous paragraph, i.e. such that $H(x_n) \not\supseteq H(x_n+)$. For $m \in M(x_n)$, we know by definition that $m = \pi(x_n+)m - \pi(x_n)m$, so if $y > x_n$, then $H(y) \subseteq H(x_n+)$, and hence m is invariant under $H(y)$, as $\pi(y)\pi(x) = \pi(x)$ if $y \geq x$. But this also means that $m = \pi(x_n+)m - \pi(x_n)m = m - \pi(x_n)m$, so $\pi(x_n)m = 0$. This shows that $M(x_n)^{G^{x_n}} = 0$, so (ii) also is verified.

To prove that the decomposition of M is unique, let $M := \bigoplus_{x \geq 0} M'(x)$ be a second decomposition with the properties (i) - (iii). Clearly, $M'(x) \neq 0$ if and only if x is one of the x_1, \dots, x_r (or $x = 0$) from above, and the same is true for $M(x)$. We know that $M'(0) = M^P = M(0)$. Inductively, assume that $M(x_i) = M'(x_i)$ for all $i < r$. Let $m \in M'(x_r)$, and write $m = m_1 + \dots + m_r$ with $m_1, \dots, m_{i-1} \in M(x_i) = M'(x_i)$ and $m_r \in M(x_r)$. We have $\pi(x_1)m = \pi(x_1)m_r = 0$, as $M(x_r)^{G^{x_r}} = M'(x_r)^{G^{x_r}} = 0$, but $\pi(x_1)m_i = m_i$ for $i < r$. It follows that $m_r = m$ and hence $M(x_r) = M'(x_r)$. This completes the proof of (a).

We already remarked that the decomposition of M only has finitely many nonzero components, so it remains to check (c) and (d). If M, N are two P -modules and $\varphi : M \rightarrow N$ an H -equivariant map, then φ restricts to $\varphi(x) : M(x) \rightarrow N(x)$. This is obvious for $x = 0$. If $x_i > 0$ is one of the finitely many elements with the property that $H(x_i) \not\supseteq H(x_i+)$, then for $m \in M(x_i)$, we immediately see that $\pi(x_i)\varphi(m) = 0$, and $\pi(x_i+)\varphi(m) = \varphi(m)$. Hence $\varphi(m) \in N(x_i)$.

This also implies (d): if $y \neq x$, then in the decomposition $M(x) = \bigoplus_{y \in \mathbb{R}_{\geq 0}} (M(x))(y)$ of the P -module $M(x)$, we have $(M(x))(y) = 0$ if $y \neq x$. Thus there are no nonzero P -morphisms $M(y) \rightarrow M(x)$, unless $y = x$.

To see that the functor $M \mapsto M(x)$ is exact, let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of P -modules. Clearly $M'(x) \subseteq M(x)$, and $M(x) \twoheadrightarrow M''(x)$, as the map $M \rightarrow M''$ is surjective, and $M(y) \rightarrow M''(x)$ is the zero map if $y \neq x$. To see that the sequence $0 \rightarrow M'(x) \rightarrow M(x) \rightarrow M''(x) \rightarrow 0$ is exact in the middle, we can use the same argument: M' maps surjectively onto the kernel of $M \rightarrow M''$. \square

Definition 4.79. If M is a P_K -module, then the decomposition from Proposition 4.77 is called the *break decomposition* of M .

Corollary 4.80. Let A be a $\mathbb{Z}[1/p]$ -algebra, and M an A -module, on which $P = P_K$ acts A -linearly through a finite quotient, i.e. the representation $P \rightarrow \text{Aut}_{\mathbb{Z}}(M)$ factors through a finite quotient and through $\text{Aut}_A(M) \subseteq \text{Aut}_{\mathbb{Z}}(M)$.

- (a) In the break decomposition $M = \bigoplus_{x \in \mathbb{R}_{\geq 0}} M(x)$, every $M(x)$ is an A -submodule of M .
- (b) If B is an A -algebra, then the break decomposition of $B \otimes_A M$ is $\bigoplus_{x \in \mathbb{R}_{\geq 0}} B \otimes_A M(x)$.

- (c) If the $\mathbb{Z}[1/p]$ -algebra A is local and noetherian, and M a free A -module of finite rank, then every $M(x)$ is free of finite rank.

Proof. (a) If $a \in A$, then multiplication by a induces a P -equivariant endomorphism of M , hence by Proposition 4.77, (c), multiplication by a maps $M(x)$ to $M(x)$ for every $x \in \mathbb{R}_{\geq 0}$.
 (b) This is clear by construction: the idempotents $\pi(x), \pi(x+)$ used in the construction of the decomposition also exist in $A[P]$ and $B[P]$.
 (c) If M is a free A -module of finite rank, then if A is noetherian, its A -submodule $M(x)$ is projective of finite rank. Hence if A is local, $M(x)$ is free of finite rank. \square

Definition 4.81. If A is a local noetherian $\mathbb{Z}[1/p]$ -algebra and M a free A -module of finite rank on which P_K acts A -linearly through a finite quotient, then the number $\text{rank}_A M(x)$ is called *multiplicity of x* . If $\text{rank}_A M(x) > 0$, then x is called *break of M* .

The real number

$$\text{Swan}(M) := \sum_{x \geq 0} x \text{rank}_A(M(x))$$

is called the *Swan conductor of M* .

- Remark 4.82.** (a) With this definition, it is obvious that $\text{Swan}(M) = 0$ if and only if the action of P_K on M is trivial, i.e. if and only if $M^{P_K} = M$.
 (b) If B is an A -algebra, then $\text{Swan}(M) = \text{Swan}(M \otimes_A B)$ by Corollary 4.80.
 (c) $\text{Swan}(M)$ is additive in short exact sequences by Proposition 4.77, (c).

Now we return to representations of the larger groups G_K^0 or G_K .

Lemma 4.83. Let M be a $\mathbb{Z}[1/p]$ -module on which G_K^0 (resp. G_K) acts such that the restricted action of P_K on M factors through a finite quotient. Then the break decomposition $M = \bigoplus_{x \geq 0} M(x)$ is a decomposition of G_K^0 -modules (resp. G_K -modules).

Proof. We only do the proof for G_K ; the argument for G_K^0 is identical. Let $H \subseteq \text{Aut}_{\mathbb{Z}}(M)$ denote the finite image of P_K , and as in the proof of Proposition 4.77, for $x > 0$ let $H(x)$ be the image of G_K^x , and for $x \geq 0$ let $H(x+)$ be the image of G_K^{x+} . We again get idempotents $\pi(x)$ and $\pi(x+)$, and $\pi(0+)$. As P_K is a normal subgroup of G_K , the elements in the image of G_K in $\text{Aut}_{\mathbb{Z}}(M)$ commute with the $\pi(x), \pi(x+)$:

$$g\pi(x+) = \left(\frac{1}{|H(x+)|} \sum_{h \in H(x+)} ghg^{-1} \right) g = \pi(x+)g,$$

and analogously for $\pi(x)$. As $M(x) = \{m \in M | \pi(x+)(1 - \pi(x))m = m\}$, the claim follows. \square

4.4.2. *Integrality.* Now let us return to ℓ -adic representations. We denote by K a complete discretely valued field with perfect residue field k of characteristic $p > 0$. We write $G_K = \text{Gal}(K^{\text{sep}}/K)$ with respect to a fixed separable closure K^{sep} of K , and $P_K = G_K^{0+}$ for its wild inertia group, see the discussion preceding Definition 4.67 and Lemma 4.75.

Let E be a finite extension of \mathbb{Q}_ℓ with ℓ a prime different from p .

Definition 4.84. If V is a finite dimensional E -vector space and $\rho : G_K \rightarrow \text{GL}(V)$ a continuous representation, then we know by Lemma 4.69 that the restriction of ρ to P_K factors through a finite group. By Lemma 4.83 and Corollary 4.80 the break decomposition

$$V = \bigoplus_{x \in \mathbb{R}_{\geq 0}} V(x)$$

is a decomposition of continuous E -representations of G_K . (For the continuity observe that ρ factors through $\bigoplus_{x \geq 0} \text{GL}(V(x))$, which carries the subspace topology in $\text{GL}(V)$).

The real number $\text{Swan}(V)$ is called the *Swan conductor* of V . By definition, $\text{Swan}(V) = 0$ if and only if V is tame.

If V is a finite dimensional E -vector space and $\rho : G_K \rightarrow \text{GL}(V)$ a continuous representation, then according to Lemma 4.66 there exists a free finite rank \mathcal{O}_E -submodule $\mathcal{V} \subseteq V$ with $\text{rank}_{\mathcal{O}_E} \mathcal{V} = \dim_E V$, such that ρ factors through $\text{GL}(\mathcal{V}) \subseteq \text{GL}(V)$. We then get an \mathcal{O}_E -linear, G_K -stable break decomposition

$$\mathcal{V} = \bigoplus_{x \geq 0} \mathcal{V}(x)$$

and Corollary 4.80 shows that $V(x) = \mathcal{V}(x) \otimes_{\mathcal{O}_E} E$. It follows that $\text{Swan}(V) = \text{Swan}(\mathcal{V})$.

Similarly, if \mathbb{F}_λ is the (finite) residue field of E and $\overline{\mathcal{V}} := \mathcal{V} \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda$ denotes the reduction of \mathcal{V} , then we obtain an \mathbb{F}_λ -linear, G_K -stable break decomposition

$$\overline{\mathcal{V}} = \bigoplus_{x \geq 0} \overline{\mathcal{V}}(x).$$

Again it follows from Corollary 4.80 that $\overline{\mathcal{V}}(x) = \mathcal{V}(x) \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda$, and hence that $\text{Swan}(V) = \text{Swan}(\mathcal{V}) = \text{Swan}(\overline{\mathcal{V}})$.

The final goal of this section is to prove that $\text{Swan}(V)$ is an integer.

Theorem 4.85. *If V is a finite dimensional E -vector space, and $\rho : G_K \rightarrow \text{GL}(V)$ a continuous representation, then $\text{Swan}(V) = b(V)$ (Definition 4.71).*

Remark 4.86. For $\text{Swan}(V)$ to be an integer, it is necessary that V really is a representation not just of P_K but (at least) of G_K^0 . For representations of P_K it can still be shown that their Swan conductors are rational numbers, see [23, p. 214].

Proof of Theorem 4.85. We compute the number $\text{Swan}(\overline{\mathcal{V}})$, where $\overline{\mathcal{V}} = \mathcal{V} \otimes_{\mathcal{O}_E} \mathbb{F}_\lambda$ with \mathcal{V} a free \mathcal{O}_E -module with continuous action $\rho_{\mathcal{O}_E} : G_K \rightarrow \text{GL}(\mathcal{V})$ such that ρ factors through $\rho_{\mathcal{O}_E}$. As the residue field \mathbb{F}_λ of E is a finite field, $\text{GL}(\overline{\mathcal{V}})$ is a finite group, and consequently G_K acts on $\overline{\mathcal{V}}$ through a finite quotient G , which corresponds to a finite Galois extension L/K . If $\overline{\mathcal{V}} = 0$ then there is nothing to do, so assume $\overline{\mathcal{V}} \neq 0$.

Let $x \in \mathbb{R}_{\geq 0}$ be an element such that $\overline{\mathcal{V}}(x) \neq 0$. Then $G^x \neq G^{x+\varepsilon}$ for all $\varepsilon > 0$ and hence $G_{[\psi_{L/K}(x)]} \neq G_{[\psi_{L/K}(x)+\varepsilon]}$ for all $\varepsilon > 0$, so $\psi_{L/K}(x) \in \mathbb{Z}_{\geq 0}$. By the computation following Definition 3.44, this means that

$$x = \varphi_{L/K}(\psi_{L/K}(x)) = \sum_{i=1}^{\psi_{L/K}(x)} \frac{|G_i|}{|G_0|}.$$

On the other hand, for an integer $i \in [1, \psi_{L/K}(x)]$ we have $G^x = G_{\psi_{L/K}(x)} \subseteq G_i$, so

$$\overline{\mathcal{V}}(x)^{G_i} \subseteq \overline{\mathcal{V}}(x)^{G^x} = 0.$$

If $i > \psi_{L/K}(x)$, then $\varphi_{L/K}(i) > x$, and hence $\overline{\mathcal{V}}(x)^{G_i} = \overline{\mathcal{V}}(x)$. Consequently we compute

$$\dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}(x)/\overline{\mathcal{V}}(x)^{G_i}) = \begin{cases} \dim_{\mathbb{F}_\lambda} \overline{\mathcal{V}}(x) & i \leq \psi_{L/K}(x) \\ 0 & i > \psi_{L/K}(x), \end{cases}$$

and thus

$$\begin{aligned} \text{Swan}(\overline{\mathcal{V}}(x)) &= x \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}(x)) \\ &= \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}(x)/\overline{\mathcal{V}}(x)^{G_i}) \end{aligned}$$

Both sides of this equation are additive with respect to direct sums, so we finally get

$$\text{Swan}(\overline{\mathcal{V}}) = \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\lambda}(\overline{\mathcal{V}}/\overline{\mathcal{V}}^{G_i}).$$

This shows that

$$\text{Swan}(\overline{\mathcal{V}}) = b(V),$$

see Proposition 4.73. In particular, note that the left-hand side and hence the right-hand side is independent of the choice of G . \square

Example 4.87. We continue Example 3.33. Consider the field $K := k((x))$, where k is an algebraically closed field of characteristic $p > 0$. Let $L := K[t]/(t^p - t - x^{-m})$. The Galois group of this extension is $G = \mathbb{F}_p$.

Assume that $(m, p) = 1$, pick a prime $\ell \neq p$, and an arbitrary nontrivial 1-dimensional representation $\chi : \mathbb{F}_p \rightarrow \mathbb{Q}_\ell^\times$. The reduction $\overline{\chi} : \mathbb{F}_p \rightarrow \mathbb{F}_\ell^\times$ is nontrivial, and we compute $\text{Swan}(\chi)$: In Example 3.33 we saw that $G_0 = G_1 = \dots = G_m \not\subseteq G_{m+1} = 0$. It follows that

$$\text{Swan}(\chi) = \text{Swan}(\overline{\chi}) = \sum_{i=1}^{\infty} \frac{|G_i|}{|G_0|} \dim_{\mathbb{F}_\ell}(\overline{\chi}/\overline{\chi}^{G_i}) = \sum_{i=1}^m 1 = m$$

5. RECOLLECTION ON ÉTALE MORPHISMS

We recall the definition of an étale morphism and state some of its properties without proof. The general references for this section are [18] and [20], see also [37, I, §3].

Definition 5.1. Let A be a ring (commutative with 1 as always). We say that an A -algebra B is *étale* (or that B is étale over A) if B is finitely presented as an A -algebra and one of the following equivalent conditions is satisfied:

- (a) B is formally étale over A , i.e. for all A -algebras C and all nilpotent ideals $I \subseteq C$ the map

$$\mathrm{Hom}_{A\text{-alg}}(B, C) \rightarrow \mathrm{Hom}_{A\text{-alg}}(B, C/I), \quad \varphi \mapsto \varphi \bmod I$$

is bijective.

- (b) For all A -algebras C and all ideals $I \subseteq C$ with $I^2 = 0$ the map

$$\mathrm{Hom}_{A\text{-alg}}(B, C) \rightarrow \mathrm{Hom}_{A\text{-alg}}(B, C/I), \quad \varphi \mapsto \varphi \bmod I$$

is bijective.

- (c) B is flat as an A -module and is unramified, i.e. $\Omega_{B/A}^1 = 0$.
 (d) B is flat as an A -module and for each prime ideal $\mathfrak{q} \subseteq B$ the natural map $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$ is a separable extension of fields, where $\mathfrak{p} := A \cap \mathfrak{q}$.
 (e) If $B = A[x_1, \dots, x_n]/I$ is a presentation of B , then for all prime ideals $\mathfrak{p} \subseteq A[x_1, \dots, x_n]$ with $\mathfrak{p} \supseteq I$, there exist polynomials $f_1, \dots, f_n \in I$ such that

$$I_{\mathfrak{p}} = (f_1, \dots, f_n) \subseteq A[x_1, \dots, x_n]_{\mathfrak{p}} \quad \text{and} \quad \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j} \notin \mathfrak{p}.$$

Definition 5.2. A morphism of schemes $f : X \rightarrow Y$ is *étale* if for any point $x \in X$ with image $y = f(x) \in Y$ there exist open neighborhoods $V = \mathrm{Spec} B$ of x and $U = \mathrm{Spec} A$ of y such that f restricts to a morphism $f|_V : V \rightarrow U$ and the induced map $A \rightarrow B$ makes B into an étale A -algebra.

Example 5.3.

- (a) Let k be a field and A a k -algebra. Then A is étale over k iff $A \cong \prod_{i \in I} L_i$, where I is some index set and L_i/k is a finite separable extension.
 (b) Open immersions are étale.
 (c) In the situation of Definition 3.11 the A -algebra B is étale iff L/K is unramified.

Proposition 5.4 ([18, (17.3.3), (17.3.4), (17.7.3)]).

- (a) The composition of étale morphisms is étale.
 (b) Let $Y' \rightarrow Y$ be a morphism of schemes. If $f : X \rightarrow Y$ is étale, then the base change map $X \times_Y Y' \rightarrow Y'$ is étale. The converse is true if $Y' \rightarrow Y$ is faithfully flat and quasi-compact (the latter condition is satisfied if e.g. Y' is noetherian.)
 (c) If $f : X \rightarrow S$ and $g : Y \rightarrow S$ are étale, so is any S -morphism $X \rightarrow Y$.

Proposition 5.5 ([20, Exp. I, Cor 5.4]). Consider a diagram

$$\begin{array}{ccc} X & \xrightarrow{f, g} & Y \\ & \searrow a & \swarrow b \\ & S, & \end{array}$$

where X, Y, S are locally noetherian and connected, b is separated and étale and $a = b \circ f = b \circ g$. Assume there exists a point $x \in X$ with $f(x) = g(x) =: y$ and that f and g induce the same morphism $k(y) \rightarrow k(x)$ on the residue fields. Then $f = g$.

6. THE ÉTALE FUNDAMENTAL GROUP

In this section let X be a connected noetherian scheme. We partly follow [20] and [48].

6.1. Definition and basic properties.

Definition 6.1. (a) An *étale covering* of X is a surjective finite étale morphism $f : Y \rightarrow X$. The category of such finite étale morphisms is denoted $\mathbf{F\acute{E}t}_X$.

- (b) A *geometric point* of X is a morphism $\bar{x} : \mathrm{Spec} \Omega \rightarrow X$, with Ω an algebraically closed field. Note that giving \bar{x} is equivalent to giving a point x of X and an embedding of the residue field $k(x)$ into Ω .
- (c) If $f : Y \rightarrow X$ is a finite étale cover and $\bar{x} : \mathrm{Spec} \Omega_1 \rightarrow X$, $\bar{y} : \mathrm{Spec} \Omega_2 \rightarrow Y$ geometric points, we say that \bar{y} *lies over* \bar{x} if there exists an algebraically closed field Ω containing both Ω_1 and Ω_2 , such that the diagram

$$\begin{array}{ccccc} \mathrm{Spec} \Omega & \longrightarrow & \mathrm{Spec} \Omega_2 & \xrightarrow{\bar{y}} & Y \\ & \searrow & & & \downarrow f \\ & & \mathrm{Spec} \Omega_1 & \xrightarrow{\bar{x}} & X \end{array}$$

commutes. For the geometric point $\mathrm{Spec} \Omega_2 \xrightarrow{\bar{y}} Y \xrightarrow{f} X$ we also write $f(\bar{y})$.

- (d) If $\bar{x} : \mathrm{Spec} \Omega \rightarrow X$ is a geometric point of X , we denote by $\mathrm{Fib}_{\bar{x}}$ the functor

$$\mathrm{Fib}_{\bar{x}} : \mathbf{F\acute{E}t}_X \rightarrow \mathbf{FiniteSet}, (Y \rightarrow X) \mapsto \mathrm{Hom}_X(\mathrm{Spec} \Omega, Y).$$

Note that $\mathrm{Fib}_{\bar{x}}(Y \rightarrow X)$ can be naturally identified with the finite set underlying the geometric fiber $Y_{\bar{x}} := Y \times_X \mathrm{Spec} \Omega$.

- (e) If \bar{x} is a geometric point of X , then the group $\mathrm{Aut}(\mathrm{Fib}_{\bar{x}})$ is called the *étale fundamental group of X with respect to the base point \bar{x}* , and it is denoted $\pi_1^{\acute{e}t}(X, \bar{x})$. When confusion is unlikely, we will also write $\pi_1(X, \bar{x})$ for the étale fundamental group.

Grothendieck defined the étale fundamental group in [20], and we will summarize its main properties in this section.

Theorem 6.2 (Grothendieck ([20, V.7])). *Let X be a connected noetherian scheme and \bar{x} a geometric point. The étale fundamental group $\pi_1^{\acute{e}t}(X, \bar{x})$ is a profinite group and the functor $\mathrm{Fib}_{\bar{x}}$ induces an equivalence*

$$\mathbf{F\acute{E}t}_X \xrightarrow{\cong} \{ \text{finite sets with continuous } \pi_1^{\acute{e}t}(X, \bar{x})\text{-action} \}.$$

In particular, open subgroups of $\pi_1^{\acute{e}t}(X, \bar{x})$ correspond to (pointed) finite étale coverings, and open normal subgroups U correspond to (pointed) Galois coverings with Galois group $\pi_1^{\acute{e}t}(X, \bar{x})/U$ (see below).

Definition 6.3. A finite étale covering $f : Y \rightarrow X$ is called *Galois covering* if Y is connected and if $\mathrm{Aut}_X(Y)$ acts transitively on $\mathrm{Fib}_{\bar{x}}(Y)$. If f is Galois, then $\mathrm{Aut}_X(Y)$ is called the *Galois group of f* .

The property of being Galois is independent of the choice of the base point by Proposition 5.5. By the same proposition $\text{Aut}_X(Y)$ is a finite group, and if Y is connected then $\text{Aut}_X(Y)$ acts transitively on $\text{Fib}_{\bar{x}}(Y)$ if and only if $|\text{Aut}_X(Y)| = \deg(Y/X)$.

Alternatively, let G be a finite group, and consider the scheme $G_X := \coprod_{g \in G} X_{(g)}$, where $X_{(g)} = X$. This is the constant X -group scheme associated with G . If $f : Y \rightarrow X$ is a finite étale covering, then a homomorphism $G \rightarrow \text{Aut}_X(Y)$ gives rise to an action $\varphi : G_X \times_X Y \rightarrow Y$. If Y is connected, then f is a Galois covering with group G if and only if the morphism

$$(\text{id}, \varphi) : G_X \times_X Y \rightarrow Y \times_X Y$$

is an isomorphism, i.e. if X is a G -torsor.

Proposition 6.4. *Let X be a connected noetherian scheme and $f : Y \rightarrow X$ a finite étale covering. There exists a Galois covering $g : P \rightarrow X$, factoring through f .*

If Y is connected then one finds g , such that any other Galois covering $g' : P' \rightarrow X$ which factors through f also factors through g . More succinctly, we have the following diagram:

$$\begin{array}{ccc} P' & \overset{\exists}{\dashrightarrow} & P \\ & \searrow & \downarrow \\ & & Y \\ & \searrow g' & \downarrow f \\ & & X \end{array} \quad \begin{array}{c} \curvearrowright \\ g \end{array}$$

Example 6.5. Let K be a field and $X = \text{Spec } K$. Giving a finite étale covering $f : Y \rightarrow X$ is equivalent to giving a finite étale K -algebra A , and setting $Y = \text{Spec } A$. The K -algebra A is a finite product of separable extensions of K , see Example 5.3. Hence Y is connected if and only if A itself is a separable field extension of K , and $f : Y \rightarrow X$ is a Galois covering if and only if A/K is a Galois extension.

To see that $\pi_1^{\text{ét}}(X, \bar{x})$ is a profinite group, Grothendieck proceeds as follows. Let Gal_X be the set of Galois coverings of X . Let I be an index set, so that we can write $\text{Gal}_X = \{f_\alpha : P_\alpha \rightarrow X \mid \alpha \in I\}$. We define an ordering on I by writing $\alpha \leq \beta$ if and only if there exists an X -morphism $P_\beta \rightarrow P_\alpha$. By Proposition 6.4 this system is directed: For $\alpha, \beta \in I$ every connected component of the fiber product $P_\alpha \times_X P_\beta$ is an étale covering of X . By Proposition 6.4 we find $f_\gamma : P_\gamma \rightarrow X$ factoring through $P_\alpha \times_X P_\beta$, so $\gamma \geq \alpha, \beta$. Now we fix arbitrary geometric points $p_\alpha \in \text{Fib}_{\bar{x}}(P_\alpha)$. By Proposition 5.5, for $\alpha \leq \beta \in I$, there exists a *unique* map $\varphi_{\beta, \alpha} : P_\beta \rightarrow P_\alpha$, mapping p_β to p_α . The system $(P_\alpha, \varphi_{\beta, \alpha})$ is a directed projective system. Note that its inverse limit exists in the category of schemes because the maps $\varphi_{\beta, \alpha}$ are finite, and hence affine ([18, Prop. 8.2.3]), but in general it is not étale over X , as it is usually not locally of finite type. Now let $f : Y \rightarrow X$ be an arbitrary finite étale covering. For every $\alpha \in I$ we obtain a map $\text{Hom}_X(P_\alpha, Y) \rightarrow \text{Fib}_{\bar{x}}(Y)$,

$g \mapsto g(p_\alpha)$. These maps give rise to a map

$$\varinjlim_I \operatorname{Hom}(P_\alpha, Y) \rightarrow \operatorname{Fib}_{\bar{x}}(Y),$$

which is easily seen to be a bijection. In Grothendieck's words, $\operatorname{Fib}_{\bar{x}}$ is *strictly pro-representable*.

Moreover, it is not difficult to see that every automorphism of $\operatorname{Fib}_{\bar{x}}$ comes from a unique automorphism of the projective system $(P_\alpha, \varphi_{\alpha, \beta})$: If $\psi \in \operatorname{Aut}(\operatorname{Fib}_{\bar{x}})$ then for every $\alpha \in I$, there exists a unique automorphism of P_α mapping p_α to $\psi(p_\alpha)$, as the P_α are Galois. It follows that

$$\pi_1^{\text{ét}}(X, \bar{x}) = \operatorname{Aut}(\operatorname{Fib}_{\bar{x}}) = \operatorname{Aut}\left(\varinjlim_I \operatorname{Hom}_X(P_\alpha, -)\right) = \varprojlim \operatorname{Aut}_X(P_\alpha)^{\text{opp}}, \quad (6.1)$$

where $(-)^{\text{opp}}$ denotes the opposite group.

Example 6.6. (a) Let us bring the above construction down to earth: Let K be a field and write $X = \operatorname{Spec} K$. Choosing a geometric point \bar{x} of $\operatorname{Spec} K$ amounts to choosing an embedding of K into an algebraically closed field Ω . Without loss of generality we take $\Omega = K^{\text{alg}}$, an algebraic closure of K . The Galois coverings P_α of X in the above construction correspond to finite Galois extensions L_α . Fixing a geometric point p_α lying over \bar{x} is equivalent to fixing a K -embedding of L_α into K^{alg} . The transition maps $\varphi_{\alpha, \beta}$ then correspond to embeddings $L_\alpha \subseteq L_\beta \subseteq K^{\text{alg}}$, so the projective system $(P_\alpha, \varphi_{\alpha, \beta})$ corresponds to the inductive system of finite Galois extension of K contained in K^{alg} .

Finally, (6.1) shows that

$$\pi_1^{\text{ét}}(X, \bar{x}) = \varprojlim \operatorname{Aut}_X(\operatorname{Spec} L_\alpha)^{\text{opp}} = \varprojlim_{\substack{L \subseteq K^{\text{alg}} \\ L/K \text{ finite Galois}}} \operatorname{Aut}_K(L) = \operatorname{Gal}(K^{\text{alg}}/K).$$

(b) If X is a connected normal noetherian scheme, write $K(X)$ for its function field, i.e. for the stalk at its generic point. Fix an algebraic closure $K(X)^{\text{alg}}$, and let $\bar{\eta} : \operatorname{Spec} K(X)^{\text{alg}} \rightarrow X$ denote the associated geometric point of X . If $K(X)^{\text{unr}}$ is the composite of all finite Galois extensions $L/K(X)$, such that the normalization of X in L is étale over X , then $K(X)^{\text{unr}}$ is an (usually infinite) Galois extension of $K(X)$ and there is a natural isomorphism

$$\operatorname{Gal}(K(X)^{\text{unr}}/K(X)) \cong \pi_1^{\text{ét}}(X, \bar{\eta}).$$

Proposition 6.7 (Dependence on base point). *Again X is noetherian and connected. If \bar{x} and \bar{x}' are two geometric points of X , then there is a continuous isomorphism*

$$\pi_1^{\text{ét}}(X, \bar{x}) \cong \pi_1^{\text{ét}}(X, \bar{x}'),$$

which is canonical up to inner automorphism.

Proving the proposition amounts to constructing an isomorphism of the inverse system $(P_\alpha, \varphi_{\alpha, \beta})$ constructed using \bar{x} and the analogous inverse system constructed using \bar{x}' , which is not too difficult.

From the Definition of the fundamental group we easily see that it is functorial for schemes equipped with a geometric point.

Proposition 6.8 (Functoriality). *If $f : X' \rightarrow X$ is a morphism of connected noetherian schemes, and $\bar{x}' : \text{Spec } \Omega \rightarrow X'$ a geometric point, then f induces a continuous homomorphism of groups*

$$\pi_1^{\text{ét}}(X', \bar{x}') \rightarrow \pi_1^{\text{ét}}(X, f\bar{x}').$$

Proof. Let $Z \rightarrow X$ be a finite étale covering. Then $Z' = Z \times_X X' \rightarrow X'$ is an étale covering, and $\text{Fib}_{\bar{x}'}(Z') = \text{Fib}_{f\bar{x}'}(Z)$. It follows that any automorphism of $\text{Fib}_{\bar{x}'}$ induces an automorphism of $\text{Fib}_{f\bar{x}'}$. \square

Example 6.9. Let X be connected and normal with function field $K(X)$. Fix an algebraic closure $K(X)^{\text{alg}}$ of $K(X)$, and write $\bar{\eta}$ for the associated geometric point. With this setup Example 6.6 shows that

$$\text{Gal}(K(X)^{\text{alg}}/K(X)) = \pi_1^{\text{ét}}(\text{Spec}(K(X)), \bar{\eta}) \rightarrow \pi_1^{\text{ét}}(X, \bar{\eta})$$

is surjective.

6.2. Complex varieties. If X is a scheme of finite type over the field of complex numbers \mathbb{C} , then to X one associates a complex analytic space X^{an} . Its underlying set is $X(\mathbb{C})$. If X is a smooth complex variety, then X^{an} is a complex manifold. In this section we summarize how the étale fundamental group of X and the fundamental group of the topological space X^{an} are related. For details and proofs we refer the reader to [20, Exp. XII].

If $f : Y \rightarrow X$ is a finite étale covering, then $f^{\text{an}} : Y^{\text{an}} \rightarrow X^{\text{an}}$ is a covering space of X^{an} with finite fibers. This defines a functor from the category $\mathbf{F\acute{E}t}_X$ to the category of covering spaces of X^{an} with finite fibers. The Riemann Existence Theorem ([20, Exp. XII, Thm. 5.1]) states that this functor is in fact an equivalence. We will use the following reformulation in terms of fundamental groups.

Theorem 6.10. *Let X be a connected finite type \mathbb{C} -scheme and $x \in X(\mathbb{C})$. Then there is a natural continuous isomorphism*

$$\pi_1^{\text{ét}}(X, x) \xrightarrow{\cong} \pi_1(X^{\text{an}}, x)^\wedge,$$

where the right-hand side is the profinite completion of the fundamental group of the topological space X^{an} , based at x .

One uses this theorem to prove:

Corollary 6.11. *Let X be a smooth complex curve of genus g . Define $U := X \setminus \{x_1, \dots, x_n\}$ as complement of finitely many closed points. If $\bar{u} \in U$ is a geometric point, then the étale fundamental group $\pi_1^{\text{ét}}(U, \bar{u})$ is isomorphic to the profinite completion of the group*

$$\langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n \mid [a_1, b_1] \cdot \dots \cdot [a_g, b_g] c_1 \cdot \dots \cdot c_n = 1 \rangle.$$

More generally the corollary is true over any algebraically closed field of characteristic 0, [20, Exp. XIII, Cor. 2.12].

Similarly, since the fundamental group of a smooth quasi-projective complex variety is a finitely generated group (this can be deduced from the curve case via Lefschetz type theorems), we obtain:

Corollary 6.12. *If X is a smooth complex quasi-projective variety, and $x \in X(\mathbb{C})$, then $\pi_1^{\text{ét}}(X, x)$ is topologically finitely generated.*

With considerable effort, the theory of specialization maps for fundamental groups allows to generalize the corollary:

Theorem 6.13. *Let k be an algebraically closed field and X a smooth quasi-projective k -variety with geometric point \bar{x} . Then $\pi_1^{\text{ét}}(X, \bar{x})$ is topologically finitely generated in each of the following two cases:*

- $\text{char}(k) = 0$ (implicit in [20, p.291], see also [28, Thm. 1]).
- $\text{char}(k) > 0$ and X is proper ([20, Exp. X, Thm. 2.9]).

The picture is different for non-proper varieties in positive characteristic:

Example 6.14 ([17]). If k is an algebraically closed field of characteristic $p > 0$, U a smooth affine curve over k and \bar{u} a geometric point of U , then the maximal pro- p -quotient of $\pi_1^{\text{ét}}(U, \bar{u})$ is a free pro- p -group on $\#k$ generators. This can be seen by cohomological methods: One checks that $H_{\text{ét}}^i(U, \mathbb{F}_p) = H^i(\pi_1^{\text{ét}}(U, \bar{u}), \mathbb{F}_p)$, for all $i \geq 0$, where the right-hand side is continuous group cohomology. Since U is affine, we have $H_{\text{ét}}^2(U, \mathbb{F}_p) = 0$. The numbers of generators and relations of the maximal pro- p -quotient of $\pi_1^{\text{ét}}(U, \bar{u})$ can be described in terms of the \mathbb{F}_p -dimension of the groups H^1 and H^2 above ([47, §4.]).

6.3. Tame coverings. The pathologies that appear in characteristic $p > 0$, exemplified by Example 6.14, can partly be remedied by restricting the attention to *tame* coverings. The following Definition is due to Kerz-Schmidt, following work of Wiesend, [27].

Definition 6.15. (a) Let X be a normal connected scheme and $U \subseteq X$ an open subset. If $f : V \rightarrow U$ is a finite étale covering, then f is said to be *tamely ramified along $X \setminus U$* , if and only if for every codimension 1 point $\eta \in X \setminus U$, the extension $K(X) \subseteq K(V)$ is tamely ramified with respect to the discrete valuation defined by η .

(b) If U is a normal connected separated scheme of finite type over a field, then a finite étale covering $f : V \rightarrow U$ is said to be *tame* if for all open immersions $U \hookrightarrow X$ with X a proper connected normal k -scheme, f is tamely ramified with respect to $X \setminus U$.

If U is not connected, then $f : V \rightarrow U$ is tame if the induced coverings of the components of U are tame.

(c) The category of tame coverings of U is denoted $\mathbf{F\acute{E}t}_U^{\text{tame}}$. It is a full subcategory of $\mathbf{F\acute{E}t}_U$.

Remark 6.16. (a) The normality assumption guarantees that the local ring $\mathcal{O}_{X, \eta}$ at a codimension 1 point is a discrete valuation ring.

(b) If U is a scheme over a field of characteristic 0, then every finite étale covering is tame.

(c) In Definition 6.15, (b), note that there always exists a normal connected *proper* k -scheme containing U : By Nagata's compactification theorem ([34], [11]) there exists a proper X containing U , which we can then normalize.

(d) In Definition 6.15, (b), if $\dim U = 1$, then there exists a *unique* normal proper curve X containing U . So in this case tameness of coverings of U only has to be checked on this single compactification.

In [20, Exp. XIII, §.2] Grothendieck defined the notion of a tame covering, but only with respect to a normal crossings divisor (see Definition 6.17 and Theorem 6.18, (c) below). In characteristic 0, the famous theorem of Hironaka on resolution of singularities states that one can always find a *smooth* proper variety X containing U , such that $X \setminus U$ is a strict normal crossings divisor. Proving the analogous statement in positive characteristic is an open problem. Theorem 6.18 below shows that Definition 6.15, (b), agrees with Grothendieck's notion, whenever a comparison is possible.

Definition 6.17. Let X be a locally noetherian scheme and D an effective Cartier divisor on X .

- (a) D is said to have *strict normal crossings* if for every $x \in \text{supp } D$ the following hold:
 - $\mathcal{O}_{X,x}$ is a regular local ring.
 - There exists a regular sequence t_1, \dots, t_n and $m \in \{1, \dots, n\}$, such that the ideal $\mathcal{O}_X(-D)_x \subseteq \mathcal{O}_{X,x}$ is generated by $t_1 \cdots t_m$. In other words: locally, D looks like the intersection of coordinate hyperplanes.
- (b) D is said to have *normal crossings*, if for every $x \in \text{supp } D$ there exists an étale morphism $f : V \rightarrow X$, such that $x \in \text{im}(f)$ and such that f^*D has strict normal crossings. In other words: étale locally, D looks like the intersection of coordinate hyperplanes.

Theorem 6.18 (Kerz-Schmidt, [27]). *Let $f : V \rightarrow U$ be a finite étale covering of regular separated schemes over a field k . The following statements are equivalent:*

- (a) f is tame.
- (b) For any regular k -curve C and any morphism $\varphi : C \rightarrow U$, the induced covering $V \times_U C \rightarrow C$ is tamely ramified along $\overline{C} \setminus C$, where \overline{C} is the unique normal proper k -curve containing C as an open subscheme.

If there exists an open immersion $U \hookrightarrow X$ with X a smooth, proper, separated, finite type k -scheme, such that $X \setminus U$ is a normal crossings divisor, then (a) and (b) are equivalent to

- (c) f is tamely ramified along $X \setminus U$.

6.4. The tame fundamental group. With essentially the same method as used in Section 6.1 one proves:

Theorem 6.19 ([27]). *Let U be a finite type, connected, separated, normal k -scheme, and \bar{u} a geometric point of U . Denote by $\text{Fib}_{\bar{u}}^{\text{tame}}$ the restriction of the functor $\text{Fib}_{\bar{u}}$ to $\mathbf{F}\acute{\text{E}}\mathbf{t}_U^{\text{tame}}$. Then the group $\pi_1^{\text{tame}}(U, \bar{u}) := \text{Aut}(\text{Fib}_{\bar{u}}^{\text{tame}})$ is profinite, and $\text{Fib}_{\bar{u}}^{\text{tame}}$ induces an equivalence*

$$\mathbf{F}\acute{\text{E}}\mathbf{t}_U^{\text{tame}} \xrightarrow{\cong} \{\text{finite sets with continuous } \pi_1^{\text{tame}}(U, \bar{u})\text{-action}\}.$$

Definition 6.20. The profinite group $\pi_1^{\text{tame}}(U, \bar{u})$ from the theorem is called the *tame fundamental group* of U with respect to the base point \bar{u} .

It is not difficult to check the following basic properties from the definitions.

Proposition 6.21. *Let k be a field and U, \bar{u} as in Theorem 6.19. Then*

- (a) *There is a canonical continuous surjective map*

$$\pi_1^{\text{ét}}(U, \bar{u}) \twoheadrightarrow \pi_1^{\text{tame}}(U, \bar{u}). \quad (6.2)$$

corresponding to the inclusion $\mathbf{F}\acute{\mathbf{E}}\mathbf{t}_U^{\text{tame}} \subseteq \mathbf{F}\acute{\mathbf{E}}\mathbf{t}_U$.

- (b) *The kernel of (6.2) can be described as follows: If X is a normal compactification of U and $\eta \in X \setminus U$ a codimension 1 point, denote by K_η^{sh} the fraction field of the strict henselisation of the discrete valuation ring $\mathcal{O}_{X, \eta}$. Fix an algebraic closure $\overline{K_\eta^{\text{sh}}}$. Then there is a morphism $G := \text{Gal}(\overline{K_\eta^{\text{sh}}}/K_\eta^{\text{sh}}) \rightarrow \pi_1^{\text{ét}}(U, \bar{u})$, canonical up to inner automorphism of $\pi_1^{\text{ét}}(U, \bar{u})$ by Proposition 6.7 and Proposition 6.8. Denote by P_η the image of the wild ramification group of G . The kernel of (6.2) is the smallest closed normal subgroup \mathbf{P} of $\pi_1^{\text{ét}}(U, \bar{u})$ containing P_η , where η runs through all codimension 1 points on all normal compactifications X of U .*
- (c) *If $\text{char } k = p \geq 0$, then the map (6.2) induces an isomorphism*

$$\pi_1^{\text{ét}}(U, \bar{u})^{(p')} \xrightarrow{\cong} \pi_1^{\text{tame}}(U, \bar{u})^{(p')}$$

where $(-)^{(p')}$ denotes the maximal pro-prime-to- p -quotient.

Proof. We sketch the argument.

- (a) As before let $\{P_\alpha | \alpha \in I\}$ be the set of Galois coverings of U . We saw above Example 6.5 that fixing points $p_\alpha \in \text{Fib}_{\bar{u}}(P_\alpha)$ defines unique morphisms $\varphi_{\beta, \alpha} : P_\beta \rightarrow P_\alpha$ for $\beta \geq \alpha$, and that the projective system $(P_\alpha, \varphi_{\beta, \alpha})$ induces an isomorphism $\pi_1^{\text{ét}}(U, \bar{u}) \cong \varprojlim_{\alpha \in I} \text{Aut}_U(P_\alpha)^{\text{opp}}$. If I^{tame} denotes the subset of I such that $\alpha \in I^{\text{tame}}$ if and only if $P_\alpha \rightarrow U$ is tame, then I^{tame} is also a directed set (exercise!), and $\pi_1^{\text{tame}}(U, \bar{u}) = \varprojlim_{\alpha \in I^{\text{tame}}} \text{Aut}_U(P_\alpha)^{\text{opp}}$. The claim follows.
- (b) Just for this proof, if G is a profinite group, we write $\mathbf{Set}(G)$ for the category of finite sets with continuous G -action. The categories $\mathbf{Set}(\pi_1^{\text{ét}}(U, \bar{u})/\mathbf{P})$ and $\mathbf{Set}(\pi_1^{\text{tame}}(U, \bar{u}))$ are strictly full subcategories of $\mathbf{Set}(\pi_1^{\text{ét}}(U, \bar{u}))$, and our goal is to show that they are identical.

In other words, we need to show that if $f : V \rightarrow U$ is a finite connected étale covering, then f is tame if and only if \mathbf{P} acts trivially on $\text{Fib}_{\bar{u}}(V)$. In fact we may assume f to be Galois étale.

Let $K(U)$, $K(V)$ denote the function fields of U and V . Note that $\text{Gal}(K(V)/K(U)) = \text{Aut}_U(V)^{\text{opp}}$ is (noncanonically) isomorphic to $\text{Fib}_{\bar{u}}(V)$ in $\mathbf{Set}(\pi_1^{\text{ét}}(U, \bar{u}))$. If \mathbf{P} acts trivially on $\text{Fib}_{\bar{u}}(V)$, then the image of the group P_η in $\text{Gal}(K(V)/K(U))$ is trivial for all codimension 1 points $\eta \in X \setminus U$ in all normal compactifications X of U . But this means that $K(V)/K(U)$ is tamely ramified at η .

Conversely, if f is tame, then $K(V)/K(U)$ is tame with respect to all codimension 1 points η in $X \setminus U$ in some normal compactification X of U . This means that the image of P_η is trivial in $\text{Gal}(K(V)/K(U))$. But $\text{Gal}(K(V)/K(U)) = \text{Aut}_U(V)^{\text{opp}}$ is a quotient of $\pi_1^{\text{ét}}(U, \bar{u})$, so the image of \mathbf{P} in $\text{Gal}(K(V)/K(U))$ is trivial, and thus \mathbf{P} acts trivially on $\text{Fib}_{\bar{u}}(V)$.

- (c) This follows from the previous part, as the groups P_η are pro- p -groups.

□

Using the fact that every curve can be lifted to characteristic 0 together with Corollary 6.11, Grothendieck proves the following structure theorem for curves.

Theorem 6.22 ([20, Exp. XIII, Cor. 2.12]). *If k is an algebraically closed field of characteristic $p \geq 0$, X a smooth projective curve of genus g over k and $U = X \setminus \{x_1, \dots, x_n\}$, with x_1, \dots, x_n closed points, then for any geometric point \bar{u} of U , the tame fundamental group $\pi_1^{\text{tame}}(U, \bar{u})$ is a quotient of the profinite completion of the group*

$$\langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n \mid [a_1, b_1] \cdot \dots \cdot [a_g, b_g] c_1 \cdot \dots \cdot c_n = 1 \rangle. \quad (6.3)$$

Moreover, the maximal prime-to- p -quotient of $\pi_1^{\text{tame}}(U, \bar{u})$ is isomorphic to the pro-prime-to- p -completion of (6.3).

7. ℓ -ADIC SHEAVES

In this section we recall the definition of an étale sheaf on a scheme as well as the notions of constructible and lisse sheaves with ℓ -adic coefficients. We show that lisse sheaves correspond to continuous representations of the fundamental group. The references for this section are [12, Arcata, II and Rappart, 1.2], [13, (1.1)] and [19, Exp. V, VI].

Convention 7.1. Throughout this section schemes are assumed to be *separated* and *noetherian*.

7.1.1. Étale neighborhoods. Let X be a scheme. Recall that a *geometric point* of X is a morphism $\bar{x} = \text{Spec } k \rightarrow X$, where k is an algebraically closed field. If x is the image of $\bar{x} \rightarrow X$, we say that \bar{x} is a geometric point over x or that x is the center of $\bar{x} \rightarrow X$. By abuse of notation we also denote by \bar{x} the morphism $\bar{x} \rightarrow X$. An *étale neighborhood* of \bar{x} is a diagram

$$\begin{array}{ccc} & U & \\ & \nearrow & \downarrow u \\ \bar{x} & \longrightarrow & X, \end{array}$$

where u is étale. A morphism between two étale neighborhoods of \bar{x} is given by the obvious commutative diagram. By Proposition 5.5 there is at most one morphism between two étale neighborhoods of \bar{x} . It follows that the opposite category of étale neighborhoods of \bar{x} is filtered.

7.1.2. Étale sheaves. Let X be a scheme. We denote by $(\text{ét}/X)$ the category with objects the étale X -schemes and morphisms the X -morphisms (these are automatically étale by Proposition 5.4). Let A be a ring. Then an *étale presheaf* of A -modules on X is a functor

$$\mathcal{F} : (\text{ét}/X)^{\text{op}} \rightarrow (A\text{-mod}).$$

By convention $F(\emptyset) = 0$. In case $A = \mathbb{Z}$, we will simply say that \mathcal{F} is an étale presheaf on X or is a presheaf on $X_{\text{ét}}$. Let $\bar{x} \rightarrow X$ be a geometric point. The stalk of a presheaf of A -modules \mathcal{F} on $X_{\text{ét}}$ at \bar{x} is given by

$$\mathcal{F}_{\bar{x}} := \varinjlim_{\bar{x} \rightarrow U} \mathcal{F}(U),$$

where the limit is over the étale neighborhoods of \bar{x} . (One can take just one neighborhood for each isomorphism class to avoid set theoretical problems.)

We say \mathcal{F} is an *étale sheaf* of A -modules on X or a *sheaf of A -modules* on $X_{\text{ét}}$, if it is a presheaf of A -modules on $X_{\text{ét}}$ and for any étale map $U \rightarrow X$ and all families $\{u_i : U_i \rightarrow U \mid i \in I\}$ of étale maps with $\bigcup_i u_i(U_i) = U$ the sequence

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j). \quad (7.1)$$

is exact. Recall that the exactness of this sequence means that (1) an element $s \in \mathcal{F}(U)$ is zero iff $s|_{U_i} = 0$ for all i and (2) for a collection of elements $s_i \in \mathcal{F}(U_i)$, $i \in I$, there exists an element $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ iff $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all $i, j \in I$. (In particular, if $\{U_i \rightarrow U\}$ is a Zariski open cover this is just the usual sheaf condition.) A morphism between (pre-)sheaves of A -modules on $X_{\text{ét}}$ is a natural transformation of functors.

If P is a presheaf of A -modules on $X_{\text{ét}}$ then there exists a sheaf of A -modules aP on $X_{\text{ét}}$ and a morphism $P \rightarrow aP$ such that any morphism $P \rightarrow \mathcal{F}$ with \mathcal{F} a sheaf on $X_{\text{ét}}$ factors uniquely as $P \rightarrow aP \rightarrow \mathcal{F}$. The sheaf aP is called the *sheaf associated to P* or the *sheafification of P* and it satisfies $(aP)_{\bar{x}} = P_{\bar{x}}$ for any geometric point $\bar{x} \rightarrow X$. The category of presheaves of A -modules on $X_{\text{ét}}$ is abelian. Hence the category of sheaves of A -modules on $X_{\text{ét}}$ is also abelian. (The cokernel of a morphism of sheaves is given by the sheafification of the cokernel in the bigger category of presheaves.) A sequence of sheaves of A -modules $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is exact if and only if the sequence of A -modules $0 \rightarrow \mathcal{F}'_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}''_{\bar{x}} \rightarrow 0$ is exact for all geometric points $\bar{x} \rightarrow X$.

Remark 7.2. Let \mathcal{F} be a presheaf on $X_{\text{ét}}$. Then \mathcal{F} is a sheaf iff the sequence (7.1) is exact in the following two situations: (1) $U \rightarrow X$ is an étale map and $U = \bigcup_i U_i$ is a Zariski open cover (i.e. the family $\{u_i\}$ is given by the open immersions $\{U_i \hookrightarrow U\}$) and (2) $U \rightarrow X$ is an étale map with U affine and the family $\{u_i\}$ consists just of one affine, surjective and étale map $U' \rightarrow U$, see e.g. [37, II, Prop. 1.5].

Example 7.3. Let X be a scheme.

- (a) Let M be an A -module. For $U \rightarrow X$ an étale map, denote by $\pi_0(U)$ the set of connected components of U (it is finite by Convention 7.1). Then $U \mapsto M^{\pi_0(U)}$ defines a sheaf of A -modules on $X_{\text{ét}}$. It is called the *constant étale sheaf associated to M* and denoted by M_X or simply by M again.
- (b) The assignment $U \mapsto \Gamma(U, \mathcal{O}_U)$ defines a sheaf of $\Gamma(X, \mathcal{O}_X)$ -modules on $X_{\text{ét}}$. (Indeed, by Remark 7.2 it suffices to show that for a faithfully flat morphism of rings $\varphi : A \rightarrow B$, the sequence $0 \rightarrow A \xrightarrow{\varphi} B \xrightarrow{d} B \otimes_A B$ is exact, where $d(b) = 1 \otimes b - b \otimes 1$. Since φ is faithfully flat this

is equivalent to proving that the sequence is exact when tensored with B . Thus we have to show that $0 \rightarrow B \xrightarrow{d_0} B^{\otimes_A 2} \xrightarrow{d_1} B^{\otimes_A 3}$ is exact, where $d_0(b) = 1 \otimes b$ and $d_1(b_1 \otimes b_2) = b_1 \otimes 1 \otimes b_2 - 1 \otimes b_1 \otimes b_2$. Define $s_0 : B^{\otimes 2} \rightarrow B$ by $s_0(b_1 \otimes b_2) = b_1 b_2$ and $s_1 : B^{\otimes 3} \rightarrow B^{\otimes 2}$, $s_1(b_1 \otimes b_2 \otimes b_3) = b_1 \otimes b_2 b_3$. Then $\text{id}_B = s_0 \circ d_0$ and $\text{id}_{B^{\otimes 2}} = d_0 \circ s_0 + s_1 \circ d_1$, hence the exactness.) The underlying sheaf of abelian groups on $X_{\text{ét}}$ is denoted by $\mathbb{G}_{a,X}$ or simply \mathbb{G}_a if it is clear that we view it as a sheaf on $X_{\text{ét}}$.

- (c) The assignment $U \rightarrow \Gamma(U, \mathcal{O}_U)^\times$ defines a sheaf on $X_{\text{ét}}$. This follows directly from b) above. We denote this sheaf by $\mathbb{G}_{m,X}$ or simply \mathbb{G}_m .
- (d) Let $n \geq 1$ be a natural number. We define the sheaf μ_n on $X_{\text{ét}}$ as the kernel of multiplication by n on \mathbb{G}_m , i.e. we have an exact sequence

$$0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m.$$

Explicitly $\mu_n(U) = \{a \in \Gamma(U, \mathcal{O}_U) \mid a^n = 1\}$. If R is a ring we also write by abuse of notation $\mu_n(R) := \{a \in R \mid a^n = 1\}$. We elaborate a bit.

- (i) Assume n is invertible in \mathcal{O}_X then

$$0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \rightarrow 0$$

is an exact sequence of sheaves on $X_{\text{ét}}$. Only the surjectivity has to be proved and this follows from the fact that for all U and all $a \in \mathbb{G}_m(U)$ the map $\text{Spec } \mathcal{O}_U[t]/(t^n - a) \rightarrow U$ is étale.

- (ii) Let R be a ring in which n is invertible and in which the polynomial $t^n - 1$ decomposes, i.e.

$$t^n - 1 = (t - \zeta_1) \cdots (t - \zeta_n) \quad \text{in } R[t]. \quad (7.2)$$

Then $\{\zeta_1, \dots, \zeta_n\}$ is a cyclic subgroup of order n of R^\times . Indeed, let K be a field and $\varphi : R \rightarrow K$ a morphism. Then φ induces a morphism $\{\zeta_1, \dots, \zeta_n\} \rightarrow \mu_n(K)$. This morphism is injective. (Else derivating (7.2) would give $n\varphi(\zeta_i)^{n-1} = 0$ for some i , which is absurd since $n, \varphi(\zeta_i) \in K^\times$.) Thus this morphism is bijective and we know that $\mu_n(K)$ is cyclic of order n .

Furthermore, any morphism $R \rightarrow A$ into a *local* ring A induces a bijection $\{\zeta_1, \dots, \zeta_n\} = \mu_n(A)$. Indeed, if A is a local R -algebra any $a \in A$ with $a^n = 1$ is of the form $a = \zeta_i + x$ for some i and $x \in \mathfrak{m}_A$ the maximal ideal of A . We obtain $1 = (\zeta_i + x)^n = 1 + xu$ with $u \in A^\times$, i.e. $x = 0$. This proves the bijection.

- (iii) Assume R is as in (ii) above and X is an R -scheme. Then the choice of a cyclic generator $\zeta \in R$ of $\{\zeta_1, \dots, \zeta_n\}$ defines an isomorphism

$$\mu_n \cong (\mathbb{Z}/n\mathbb{Z})_X \quad \text{on } X_{\text{ét}}.$$

Explicitly for all étale maps $U \rightarrow X$ we have an isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^{\pi_0(U)} \xrightarrow{\sim} \mu_n(U), \quad (\bar{m}_i)_{i \in \pi_0(U)} \mapsto (\zeta^{m_i})_{i \in \pi_0(U)}.$$

It suffices to prove this Zariski (or even étale) locally and hence it follows from (ii) above.

Notice that the statements (i)- (iii) are wrong if n is not invertible on X .

7.1.3. Direct and inverse image. Let $\pi : X \rightarrow Y$ be a morphism of schemes and A a ring. If \mathcal{F} is a sheaf of A -modules on $X_{\text{ét}}$, then we obtain a sheaf of A -modules $\pi_*\mathcal{F}$ on $Y_{\text{ét}}$ via

$$\pi_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X), \quad V \rightarrow Y \text{ étale.}$$

We call this sheaf the *direct image of \mathcal{F} under π* . We get a left exact functor from the category of A -modules on $X_{\text{ét}}$ to the category of A -modules on $Y_{\text{ét}}$, $\mathcal{F} \mapsto \pi_*(\mathcal{F})$. This functor has a left adjoint denoted by π^* , i.e. for each \mathcal{F} on $X_{\text{ét}}$ and \mathcal{G} on $Y_{\text{ét}}$ there is an isomorphism of abelian groups

$$\text{Hom}(\mathcal{G}, \pi_*\mathcal{F}) \cong \text{Hom}(\pi^*\mathcal{G}, \mathcal{F}),$$

which is functorial in \mathcal{F} and \mathcal{G} . We call $\pi^*\mathcal{G}$ the *inverse image of \mathcal{G} under π* . If \bar{x} is a geometric point on X and we denote $\pi(\bar{x})$ the geometric point of Y given by the composition $\bar{x} \rightarrow X \rightarrow Y$, then we have

$$(\pi^*\mathcal{G})_{\bar{x}} = \mathcal{G}_{\pi(\bar{x})}. \quad (7.3)$$

It follows that π^* is an exact functor. By abuse of notation we also write $\mathcal{G}|_X := \pi^*\mathcal{G}$. In case π is étale $\pi^*\mathcal{G}$ coincides with the restriction of the functor \mathcal{G} to the category ($\text{ét}/X$).

Example 7.4. (a) Let $\pi : X \rightarrow Y$ be a morphism of schemes and M an A -module. Then (with the notation from Example 7.3, (a))

$$\pi^*(M_Y) = M_X.$$

Indeed, for an étale map $V \rightarrow Y$ there is a natural morphism

$$M_Y(V) = M^{\pi_0(V)} \rightarrow M^{\pi_0(V \times_Y X)} = \pi_*(M_X)(V).$$

(If V is connected it is the diagonal.) This induces a morphism of sheaves $M_Y \rightarrow \pi_*(M_X)$ on $Y_{\text{ét}}$, by adjunction also a morphism $\pi^*(M_Y) \rightarrow M_X$. It suffices to check that this latter map is an isomorphism on the stalks at the geometric points of X . This follows directly from (7.3).

(b) In general $\pi_*(M_X)$ is not isomorphic to M_Y . For example, if $i : Z \hookrightarrow X$ is a closed immersion then $i_*(M_Z)$ is zero in the stalks at geometric points whose center lies in $X \setminus Z$.

7.1.4. Action of a finite group. Let G be a finite group acting on a scheme X , i.e. we are given a group homomorphism $G \rightarrow \text{Aut}(X)$. Assume G acts admissibly on X , i.e. X is a union of open affines $U = \text{Spec } A$ such that the action of G restricts to an action on U . (This is e.g. the case if X is quasi-projective over an affine scheme). Then we can form the quotient $\pi : X \rightarrow X/G$, where X/G is defined by glueing the schemes $U/G := \text{Spec } A^G$, for U as above. We have $\text{Hom}(X, Y)^G = \text{Hom}(X/G, Y)$ for all schemes Y , where G acts on $\text{Hom}(X, Y)$ via precomposition, see [20, Exp.V, 1.].

An étale sheaf of A -modules on (X, G) is an étale sheaf of A -modules \mathcal{F} on X together with morphisms

$$\mathcal{F}(\sigma) : \mathcal{F} \rightarrow \sigma^*\mathcal{F}, \quad \sigma \in G,$$

such that $\mathcal{F}(1_G) = \text{id}_{\mathcal{F}}$ and $\mathcal{F}(\tau\sigma) = \tau^*(\mathcal{F}(\sigma)) \circ \mathcal{F}(\tau)$. Sometimes we just say \mathcal{F} is a sheaf with G -action on X and it is understood that this action is compatible with the given action on X . For any geometric point $\bar{x} \rightarrow X$ we obtain isomorphisms $\mathcal{F}_{\bar{x}} \xrightarrow{\cong} \mathcal{F}_{\sigma(\bar{x})}$, where $\sigma(\bar{x})$ denotes the composition $\bar{x} \rightarrow X \xrightarrow{\sigma} X$. Also notice that we have natural isomorphisms $\sigma_*\sigma^*\mathcal{F} \cong \mathcal{F} \cong \sigma^*\sigma_*\mathcal{F}$ induced by adjunction; hence to give a map $\mathcal{F}(\sigma)$ as above is equivalent to give a map $\sigma_*\mathcal{F} \rightarrow \mathcal{F}$.

Let $\pi : X \rightarrow X/G$ be as above. The action of G on \mathcal{F} induces maps $\sigma_*\mathcal{F} \rightarrow \mathcal{F}$ and applying π_* we obtain a map $\pi_*\mathcal{F} = \pi_*\sigma_*\mathcal{F} \rightarrow \pi_*\mathcal{F}$. Thus G acts on $\pi_*\mathcal{F}$ (where we equip X/G with the trivial G -action). We can therefore form the sheaf of G -invariant elements $(\pi_*\mathcal{F})^G$ whose section on $U \rightarrow X/G$ (étale map) are given by

$$(\pi_*\mathcal{F})^G(U) = \{a \in \mathcal{F}(U \times_{X/G} X) \mid \sigma \cdot a = a \text{ for all } \sigma \in G\},$$

where we abbreviate the notation $\sigma \cdot a := \mathcal{F}(\sigma)(a)$.

7.1.5. Extension by zero. Let $j : U \hookrightarrow X$ be an open immersion and denote by $i : Z \hookrightarrow X$ the closed immersion of the complement of U (equipped with some scheme structure). Let \mathcal{F} be a sheaf of A -modules on U . Then we define the *extension by zero* of \mathcal{F} as

$$j_!\mathcal{F} := \text{Ker}(j_*\mathcal{F} \rightarrow i_*i^*j_*\mathcal{F}),$$

where $j_*\mathcal{F} \rightarrow i_*i^*j_*\mathcal{F}$ is the adjunction map for (i_*, i^*) . For a point $x \in X$ and \bar{x} a geometric point over x we have

$$(j_!\mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}}, & \text{if } x \in U, \\ 0, & \text{if } x \in Z. \end{cases}$$

It follows that $j_!$ is an exact functor from the category A -modules on $U_{\text{ét}}$ to the category of A -modules on $X_{\text{ét}}$. It is left adjoint to j^* i.e. there is a functorial isomorphism

$$\text{Hom}(j_!\mathcal{G}, \mathcal{F}) \cong \text{Hom}(\mathcal{G}, j^*\mathcal{F}),$$

where \mathcal{G} is a sheaf on $U_{\text{ét}}$ and \mathcal{F} a sheaf on $X_{\text{ét}}$.

Definition 7.5. Let X be a scheme, A a noetherian ring and M an A -module. Then we call an étale sheaf \mathcal{F} of A -modules on X *locally constant with stalk M* if there exists a family $\{u_i : U_i \rightarrow X\}$ of étale maps with $\bigcup_i u_i(U) = X$ such that $\mathcal{F}|_{U_i} = M_{U_i}$ for all i .

Proposition 7.6. *In the situation of Definition 7.5 assume M is finite (as a set). Then \mathcal{F} is represented by a finite étale group scheme over X , i.e. there is a finite étale morphism $X_{\mathcal{F}} \rightarrow X$ such that*

$$\mathcal{F} \cong \text{Hom}_X(-, X_{\mathcal{F}}) \quad \text{on } (\text{ét}/X).$$

Assume X is connected, then it follows that there exists a connected finite étale Galois cover $P \rightarrow X$ which trivializes \mathcal{F} , i.e. $\mathcal{F}|_P \cong M_P$.

Before we prove the proposition we recall the following statement from descent theory:

Theorem 7.7 ([20, VIII, Thm. 2.1, Cor. 5.7] and [18, Cor. (17.7.3)]). *Let $f : U \rightarrow X$ be faithfully flat and quasi-compact. Let $V \rightarrow U$ be an affine morphism such that there exists an isomorphism $\gamma : p_1^*V \xrightarrow{\sim} p_2^*V$ satisfying*

$$p_{13}^*(\gamma) \circ p_{12}^*(\gamma) = p_{23}^*(\gamma),$$

*where $p_i : U \times_X U \rightarrow U$ and $p_{ij} : U \times_X U \times_X U \rightarrow U \times_X U$ denote the projection on the respective factors, p_i^*V denotes the pullback of V along p_i and $p_{ij}^*(\gamma)$ denotes the pullback of γ along p_{ij} .*

Then there exists an affine morphism $Y \rightarrow X$ such that $V = Y \times_X U$. (One says V/U descends to Y/X .) Furthermore if $V \rightarrow U$ is finite or étale, then so is $Y \rightarrow X$.

Proof of Proposition 7.6. We find a finite family $\{u_i : U_i \rightarrow X\}$ of étale maps such that $u_i^*\mathcal{F} \cong M_{U_i}$. The constant sheaf M_{U_i} is represented by the group scheme $U_i \times M$ (= disjoint union over $|M|$ copies of U_i with group action induced by the one on M). Then $U_i \times M \rightarrow U_i$ is a finite étale U_i -group scheme representing $u_i^*\mathcal{F}$. Set $U := \bigsqcup_i U_i$. Then we have an étale and surjective (in particular faithfully flat) morphism $U \rightarrow X$ and a finite étale U -group scheme $U \times M$ representing $\mathcal{F}|_U$. Denote by $p_1, p_2 : U \times_X U \rightarrow U$ the two projections. The natural isomorphism $p_1^*(\mathcal{F}|_U) \cong p_2^*(\mathcal{F}|_U)$ induces a gluing data $p_1^*(U \times M) \cong p_2^*(U \times M)$ and hence the finite étale U -group scheme $U \times M$ descends to a finite étale map $X_{\mathcal{F}} \rightarrow X$, see Theorem 7.7. By construction and the sheaf property $X_{\mathcal{F}}$ represents \mathcal{F} over X .

If X is connected, then any connected component of $X_{\mathcal{F}}$ is still a finite étale covering of X . We take a finite étale Galois covering $P \rightarrow X$ which factors over all the connected components of $X_{\mathcal{F}}$; this can be achieved as in the discussion below Example 6.5. Then $X_{\mathcal{F}} \times_X P \cong P \times M$ represents $\mathcal{F}|_P$, i.e. $\mathcal{F}|_P \cong M_P$. \square

7.1.6. Constructible sheaves. Let X be a scheme. A subset $Z \subseteq X$ is called *locally closed* if it is the intersection of an open and a closed subset in X . We equip such a Z with the reduced scheme structure and obtain an immersion $i : Z \hookrightarrow X$.

Let A be a noetherian ring which is torsion (i.e. $mA = 0$ for some natural number m) and \mathcal{F} a sheaf of A -modules on $X_{\text{ét}}$. Then we say that \mathcal{F} is *constructible* if there exist finite type A -modules M_1, \dots, M_n and locally closed subsets $X_1, \dots, X_n \subseteq X$ such that

$$X = \bigsqcup_i X_i \quad \text{and} \quad \mathcal{F}|_{X_i} \text{ is locally constant with stalks } M_i.$$

The category of constructible A -modules is abelian, see [5, IX, Prop. 2.6].

Remark 7.8. Let $\pi : X \rightarrow Y$ be a morphism and \mathcal{G} a constructible sheaf of A -modules on $Y_{\text{ét}}$, then $\pi^*\mathcal{G}$ is a constructible A -module on $X_{\text{ét}}$. (This follows directly from Example 7.4.) If π is proper and \mathcal{F} is a constructible A -module on $X_{\text{ét}}$, then $\pi_*\mathcal{F}$ is a constructible A -module on $Y_{\text{ét}}$, see [5, Exp. XIV, Thm 1.1].

Definition 7.9. Let R be a complete local DVR with maximal ideal \mathfrak{m} . Assume R/\mathfrak{m} has characteristic $\ell > 0$. Let X be a scheme.

- (a) A *constructible R -sheaf* on X is a projective system of R -modules $\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$ on $X_{\text{ét}}$ satisfying the following two properties:
 - (a) $\mathfrak{m}^n \cdot \mathcal{F}_n = 0$ and \mathcal{F}_n is a constructible R/\mathfrak{m}^n -module on $X_{\text{ét}}$.
 - (b) $\mathcal{F}_n = \mathcal{F}_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n$, for all $n \geq 1$.
- (b) A *lisse R -sheaf* on X is a constructible R -sheaf $\mathcal{F} = (\mathcal{F}_n)$ such that each \mathcal{F}_n is a locally constant sheaf of R/\mathfrak{m}^n -modules.

Example 7.10. (a) Let R be as above and \mathcal{F} a locally constant sheaf of finitely generated R/\mathfrak{m}^{n_0} -modules on $X_{\text{ét}}$. Then we can view \mathcal{F} also as a lisse R -sheaf (\mathcal{F}_n) , via $\mathcal{F}_n := \mathcal{F}$ for $n > n_0$ and $\mathcal{F}_n := \mathcal{F} \otimes_{R/\mathfrak{m}^{n_0}} R/\mathfrak{m}^n R$, for $n \leq n_0$.

(b) Let $\pi : X \rightarrow Y$ be a proper morphism and $\mathcal{F} = (\mathcal{F}_n)$ a constructible R -sheaf on X . Then $\pi_* \mathcal{F} := (\pi_* \mathcal{F}_n)$ is a constructible R -sheaf on Y . This follows from Remark 7.8.

(c) Let ℓ be an invertible prime on X . We have surjections of \mathbb{Z}_ℓ -sheaves on $X_{\text{ét}}$, $\mu_{\ell^{n+1}} \rightarrow \mu_{\ell^n}$, $a \mapsto a^\ell$. The corresponding projective system of \mathbb{Z}_ℓ -modules on $X_{\text{ét}}$ is denoted by $\mathbb{Z}_\ell(1)$,

$$\mathbb{Z}_\ell(1) := (\mu_{\ell^n}).$$

It follows from Example 7.3, (d), (iii), that $\mathbb{Z}_\ell(1)$ is a lisse \mathbb{Z}_ℓ -sheaf on X . But notice that in general there is no étale covering $\{U_i \rightarrow X\}$ such that $\mu_{\ell^n}|_{U_i} = (\mathbb{Z}/\ell^n)_{U_i}$ for all n (e.g. $X = \text{Spec } \mathbb{Q}$). This shows that lisse sheaves are in general not locally constant projective systems of sheaves.

7.1.7. The category of lisse R -sheaves. Let R, \mathfrak{m}, ℓ be as in Definition 7.9 and X a scheme. Let $\mathcal{F} = (\mathcal{F}_n), \mathcal{G} = (\mathcal{G}_n)$ be two constructible R -sheaves on X . Then we have morphisms of R -modules

$$\text{Hom}(\mathcal{F}_{n+1}, \mathcal{G}_{n+1}) \rightarrow \text{Hom}(\mathcal{F}_n, \mathcal{G}_n), \quad \varphi \mapsto \varphi \otimes R/\mathfrak{m}^n.$$

The category of constructible (resp. lisse) R -sheaves is the category with morphisms given by

$$\text{Hom}(\mathcal{F}, \mathcal{G}) = \varprojlim_n \text{Hom}(\mathcal{F}_n, \mathcal{G}_n).$$

It is an abelian category, see [19, V, Thm 5.2.3].

7.1.8. The category of $\bar{\mathbb{Q}}_\ell$ -sheaves. Let X be a scheme and ℓ an invertible prime on X . Fix an algebraic closure $\bar{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ .

- (a) Let E be a *finite* field extension of \mathbb{Q}_ℓ inside $\bar{\mathbb{Q}}_\ell$ and R the integral closure of \mathbb{Z}_ℓ in E . The category of constructible E -sheaves on X is by definition the localization of the category of constructible R -sheaves localized with respect to the full-subcategory of torsion sheaves. This means the following:

Objects: There is an essentially surjective functor

$$(\text{constr. } R\text{-sheaves}) \rightarrow (\text{constr. } E\text{-sheaves}), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R E$$

Morphisms: $\text{Hom}(\mathcal{F} \otimes_R E, \mathcal{G} \otimes_R E) = \text{Hom}(\mathcal{F}, \mathcal{G}) \otimes_R E$.

We say that a constructible E -sheaf $\mathcal{F} \otimes_R E$ is *lisse* if there exists an étale covering $\{U_i \rightarrow X\}$ and lisse R -sheaves \mathcal{F}_i on U_i such that $\mathcal{F}|_{U_i} \otimes_R E \cong \mathcal{F}_i \otimes_R E$.

- (b) If $E'/E/\mathbb{Q}_\ell$ are finite extensions inside $\bar{\mathbb{Q}}_\ell$, then there is a natural functor (constr. E -sheaves) \rightarrow (constr. E' -sheaves) on objects given by

$$\mathcal{F} \otimes_R E \mapsto (\mathcal{F} \otimes_R E) \otimes_E E' = \mathcal{F} \otimes_{R'} E'.$$

The category of constructible $\bar{\mathbb{Q}}_\ell$ -sheaves is the inductive 2-limit over the categories of constructible E -sheaves, $E \subseteq \bar{\mathbb{Q}}_\ell$. This means the following:

Objects: For all finite extensions E/\mathbb{Q}_ℓ inside $\bar{\mathbb{Q}}_\ell$ there are functors (constr. E -sheaves) \rightarrow (constr. $\bar{\mathbb{Q}}_\ell$ -sheaves), $\mathcal{F} \mapsto \mathcal{F} \otimes_E \bar{\mathbb{Q}}_\ell$.

These satisfy:

- (a) Each object in (constr. $\bar{\mathbb{Q}}_\ell$ -sheaves) is isomorphic to an object of the form $\mathcal{F} \otimes_E \bar{\mathbb{Q}}_\ell$, for some constructible E -sheaf \mathcal{F} .
- (b) If E'/E is a finite extension and $\mathcal{F} \in$ (constr. E -sheaves), then there is a canonical isomorphism

$$\mathcal{F} \otimes_E \bar{\mathbb{Q}}_\ell \cong (\mathcal{F} \otimes_E E') \otimes_{E'} \bar{\mathbb{Q}}_\ell.$$

Morphisms: For $\mathcal{F}, \mathcal{G} \in$ (constr. E -sheaves) we have

$$\mathrm{Hom}(\mathcal{F} \otimes_E \bar{\mathbb{Q}}_\ell, \mathcal{G} \otimes_E \bar{\mathbb{Q}}_\ell) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes_E \bar{\mathbb{Q}}_\ell.$$

A *lisse $\bar{\mathbb{Q}}_\ell$ -sheaf* on X is a constructible $\bar{\mathbb{Q}}_\ell$ -sheaf which is locally of the form $\mathcal{F} \otimes_E \bar{\mathbb{Q}}_\ell$, with \mathcal{F} a lisse E -sheaf.

Definition 7.11. Let X be a scheme and ℓ an invertible prime on X .

- (a) Let R be a complete local DVR with maximal ideal \mathfrak{m} and residue characteristic $\ell > 0$. We say that a lisse R -sheaf $\mathcal{F} = (\mathcal{F}_n)$ is *constant* or *trivial* if there exists a finitely generated R -module M such that we have an isomorphism of projective systems $(\mathcal{F}_n) \cong ((M \otimes_R R/\mathfrak{m}^n)_X)$, where the index X denotes the constant sheaf associated to the module, see Example 7.3, (a). In this case we also write $\mathcal{F} = M_X$ or just $\mathcal{F} = M$.
- (b) Let E be a finite extension of \mathbb{Q}_ℓ with ring of integers R . We say that a lisse E -sheaf \mathcal{F} is *constant* or *trivial* if there exists a finite dimensional E -vector space V and an R -lattice $M \subseteq V$ such that $\mathcal{F} = M_X \otimes_R E$ in the above notations. As lisse E -sheaf $M_X \otimes_R E$ depends (up to isomorphism) only on V . We write $\mathcal{F} = V_X$ or just $\mathcal{F} = V$.
- (c) We say that a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf is *constant* or *trivial* if there exists a finite dimensional $\bar{\mathbb{Q}}_\ell$ -vector space V , a finite extension E/\mathbb{Q}_ℓ and a finite dimensional E -vector space V_E such that $V = V_E \otimes_E \bar{\mathbb{Q}}_\ell$ and $\mathcal{F} = V_{E,X} \otimes_E \bar{\mathbb{Q}}_\ell$. This is independent of the choice of E and V_E . We write $\mathcal{F} = V_X$ or just $\mathcal{F} = V$.

Convention 7.12. Let X be a scheme and assume ℓ is a prime number which is invertible on X . We say that a ring A is an ℓ -adic coefficient ring if it is either $\bar{\mathbb{Q}}_\ell$, a finite extension of \mathbb{Q}_ℓ or equal to R or R/\mathfrak{m}^n , $n \geq 1$, where R is a complete local DVR finite over \mathbb{Z}_ℓ with maximal ideal \mathfrak{m} . We can therefore speak about constructible (resp. lisse, resp. constant) A -sheaves on X . (In case $A = R/\mathfrak{m}^n$ a lisse A -sheaf \mathcal{F} is just a locally constant and

constructible sheaf of A -modules on $X_{\text{ét}}$; by Example 7.10, (a), we can view \mathcal{F} also as a lisse R -sheaf.) We can write any constructible A -sheaf in the form $(\mathcal{F}_n) \otimes_R A$ with R as above and (\mathcal{F}_n) a constructible R -sheaf.

7.1.9. *Tate twist.* Let X be a scheme. For $n \geq 1$ and $i \geq 0$ denote by

$$\mathbb{Z}/\ell^n(i)$$

the sheaf on $X_{\text{ét}}$ associated to

$$U \mapsto \underbrace{\mu_{\ell^n}(U) \otimes_{\mathbb{Z}/\ell^n} \dots \otimes_{\mathbb{Z}/\ell^n} \mu_{\ell^n}(U)}_{i\text{-times}}$$

and by

$$\mathbb{Z}/\ell^n(-i)$$

the sheaf on $X_{\text{ét}}$ associated to

$$U \mapsto \text{Hom}_U(\mathbb{Z}/\ell^n(i)|_U, \mathbb{Z}/\ell^n|_U).$$

Let A be an ℓ -adic coefficient ring and \mathcal{F} a constructible A -sheaf. Then we define the i -th Tate twist of \mathcal{F}

$$\mathcal{F}(i), \quad i \in \mathbb{Z},$$

as follows: Take a DVR R finite over \mathbb{Z}_ℓ and a constructible R -sheaf $(\mathcal{F}_n)_n$ such that $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$. Notice that the \mathcal{F}_n are in particular sheaves of \mathbb{Z}/ℓ^n -modules on $X_{\text{ét}}$. We define $\mathcal{F}_n(i)$ as the sheaf on $X_{\text{ét}}$ associated to $U \mapsto \mathcal{F}_n(U) \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^n(i)(U)$ and define

$$\mathcal{F}(i) := (\mathcal{F}_n(i)) \otimes_R A.$$

This does not depend on the choice of R and (\mathcal{F}_n) . It is a constructible A -sheaf and it is lisse if \mathcal{F} is. We have $\mathcal{F}(i)(j) = \mathcal{F}(i+j)$.

7.1.10. *Stalk of a lisse A -sheaf.* Let $\bar{x} \rightarrow X$ be a geometric point of X and ℓ a prime number which is invertible on X .

- (a) Let R be a complete local DVR with finite residue field of characteristic ℓ and $\mathcal{F} = (\mathcal{F}_n)$ a lisse R -sheaf on X . We define the stalk of \mathcal{F} at \bar{x} to be

$$\mathcal{F}_{\bar{x}} := \varprojlim_n \mathcal{F}_{n, \bar{x}}.$$

- (b) Let E be a finite extension of \mathbb{Q}_ℓ and \mathcal{F} a lisse E -sheaf. We define the stalk of \mathcal{F} at \bar{x} to be

$$\mathcal{F}_{\bar{x}} := (\varprojlim_n \mathcal{F}'_{n, \bar{x}}) \otimes_R E,$$

where R is the ring of integers of E over \mathbb{Q}_ℓ and $\mathcal{F}' = (\mathcal{F}'_n)$ is a lisse R -sheaf defined on some étale neighborhood of \bar{x} such that locally around \bar{x} we have $\mathcal{F} = \mathcal{F}' \otimes_R E$. This definition is independent of the choice of \mathcal{F}' (up to isomorphism).

- (c) Let \mathcal{F} be a lisse \mathbb{Q}_ℓ -sheaf. We define the stalk of \mathcal{F} at \bar{x} to be

$$\mathcal{F}_{\bar{x}} := (\varprojlim_n \mathcal{F}'_{n, \bar{x}}) \otimes_R \bar{\mathbb{Q}}_\ell,$$

where R is the ring of integers of a finite extension E over \mathbb{Q}_ℓ and $\mathcal{F}' = (\mathcal{F}'_n)$ is a lisse R -sheaf defined on some étale neighborhood of \bar{x} such

that locally around \bar{x} we have $\mathcal{F} = (\mathcal{F}' \otimes_R E) \otimes_E \bar{\mathbb{Q}}_\ell$. This definition is independent of the choice of E and \mathcal{F}' (up to isomorphism).

Thus, all together, if A is an ℓ -adic coefficient ring the stalk $\mathcal{F}_{\bar{x}}$ of a lisse A -sheaf \mathcal{F} at a geometric point \bar{x} is defined. Furthermore (with the above notation) $\varprojlim_n \mathcal{F}'_{n,\bar{x}}$ is a finite type R -module. (In fact it is separated for the \mathfrak{m} -adic topology and R is an \mathfrak{m} -adically complete ring, thus it is generated by any lift of a system of generators of $(\varprojlim_n \mathcal{F}'_{n,\bar{x}})/\mathfrak{m} = \mathcal{F}'_{1,\bar{x}}$, see e.g. [36, Thm 8.4].) Hence $\mathcal{F}_{\bar{x}}$ is a finite type A -module for general A .

We say that a lisse A -sheaf \mathcal{F} is *free* if its stalks are free A -modules. In this case and if X is connected the rank of \mathcal{F} – denoted by $\text{rk}(\mathcal{F})$ – is by definition the rank of $\mathcal{F}_{\bar{x}}$ at some (and hence any) geometric point of X .

7.1.11. ℓ -adic representations. Let A be an ℓ -adic coefficient ring (Convention 7.12), which is *not* $\bar{\mathbb{Q}}_\ell$. Let M be a finitely generated A -module, then we can equip it with the ℓ -adic topology as follows: There is a DVR R , finite over \mathbb{Z}_ℓ , such that A is either the fraction field or a quotient of R . We can find a finitely generated R -module M' such that $M = M' \otimes_R A$ and we define the topology on M in such a way that the image of $\{m + \ell^n \cdot M'\}_{n \geq 0}$ in M is a system of open neighborhoods of $m \in M$. This topology is independent of the choice of M' . (Indeed, if A is a finite ring this is just the discrete topology and otherwise M' is an R -lattice in M and for any other R -lattice M'' we find integers $a, b \geq 1$ such that $\ell^a \cdot M' \subseteq M''$ and $\ell^b \cdot M'' \subseteq M'$; hence M' and M'' define the same topology.) In particular the A -module of A -linear endomorphisms $\text{End}_A(M)$ also carries the ℓ -adic topology and the composition of endomorphisms is a continuous operation. Now let $\text{Aut}_A(M)$ be the group of A -linear automorphisms $\text{Aut}_A(M)$. We have an inclusion $\text{Aut}_A(M) \rightarrow \text{End}_A(M) \times \text{End}_A(M)$, $\sigma \mapsto (\sigma, \sigma^{-1})$. We put the product topology on $\text{End}_A(M) \times \text{End}_A(M)$ and the subspace topology on $\text{Aut}_A(M)$. In this way $\text{Aut}_A(M)$ becomes a topological group (compare to Definition 4.65).

Let X be a connected scheme and $\bar{x} \rightarrow X$ a geometric point. We define an A -representation of $\pi_1(X, \bar{x})$ to be a continuous group homomorphism

$$\pi_1(X, \bar{x}) \rightarrow \text{Aut}_A(M),$$

where M is a finitely generated A -module.

By definition, a $\bar{\mathbb{Q}}_\ell$ -representation of $\pi_1(X, \bar{x})$ is a group homomorphism $\rho_V : \pi_1(X, \bar{x}) \rightarrow \text{Aut}_{\bar{\mathbb{Q}}_\ell}(V)$, where V is a finite dimensional $\bar{\mathbb{Q}}_\ell$ -vector space, such that there exists a finite field extension E/\mathbb{Q}_ℓ , a finite dimensional E -vector space V_0 , an isomorphism $V \cong V_0 \otimes_E \bar{\mathbb{Q}}_\ell$ and an E -representation $\rho_{V_0} : \pi_1(X, \bar{x}) \rightarrow \text{Aut}_E(V_0)$ such that ρ_V is equal to the composition

$$\rho_V : \pi_1(X, \bar{x}) \xrightarrow{\rho_{V_0}} \text{Aut}_E(V_0) \xrightarrow{\otimes_E \bar{\mathbb{Q}}_\ell} \text{Aut}_{\bar{\mathbb{Q}}_\ell}(V).$$

Theorem 7.13. *Let X be a connected scheme and \bar{x} a geometric point. Let A be an ℓ -adic coefficient ring. Then there is a natural equivalence of categories*

$$(\text{lisse } A\text{-sheaves on } X) \xrightarrow{\sim} (A\text{-representations of } \pi_1(X, \bar{x})), \quad \mathcal{F} \mapsto \mathcal{F}_{\bar{x}}.$$

Furthermore, let $\pi : X' \rightarrow X$ be a finite étale Galois cover with Galois group G . Then the above equivalence induces an equivalence between the following

subcategories

(*lisse* A -sheaves on X , constant on X') $\xrightarrow{\sim}$ (finitely gen. $A[G]$ -modules).

If A is finite then under this equivalence a finitely generated $A[G]$ -module M corresponds to the *lisse* A -sheaf given by $(\pi_* M_{X'})^G$ (see 7.1.4 and (7.4) below for how we view $M_{X'}$ as a sheaf on (X', G)).

If A is infinite and if M is a finitely generated $A[G]$ -module, then there exists a DVR R , finite over \mathbb{Z}_ℓ and contained in A , and a finitely generated $R[G]$ -module N such that $M = N \otimes_R A$. In this case M corresponds to the *lisse* A -sheaf $((\pi_* N_{X'})^G \otimes_R R/\mathfrak{m}^n)_n \otimes_R A$.

Proof. Let (P_α) be an inverse system of finite étale Galois coverings of X with maps $\bar{x} \rightarrow P_\alpha$ making the diagram

$$\begin{array}{ccc} & & P_\alpha \\ & \nearrow & \downarrow \\ \bar{x} & \longrightarrow & X, \end{array}$$

commutative and such that an element of $\pi_1(X, \bar{x})$ is the same as a compatible system of automorphisms of (P_α) fixing \bar{x} , i.e. such that $\pi_1^{\text{ét}}(X, \bar{x}) = \varprojlim \text{Aut}(P_\alpha)^{\text{opp}}$ (see (6.1)).

First case: A is a finite ring. Let \mathcal{F} be a *lisse* A -sheaf. Given an étale neighborhood $U \rightarrow X$ of \bar{x} we set $U_\alpha := U \times_X P_\alpha$ and obtain an inverse system (U_α) of étale neighborhoods of \bar{x} . An element $\sigma \in \pi_1(X, \bar{x})$ induces a compatible system of automorphisms of (U_α) . Therefore such an element induces a map

$$\mathcal{F}(U) \rightarrow \mathcal{F}(U_\alpha) \xrightarrow{\sigma^*} \mathcal{F}(U_\alpha) \rightarrow \mathcal{F}_{\bar{x}}.$$

Taking the limit over all étale neighborhoods U of \bar{x} we get a morphism $\sigma^* : \mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}}$. We obtain in this way the structure of a $\pi_1(X, \bar{x})$ -representation on $\mathcal{F}_{\bar{x}}$. Clearly this construction is functorial.

For the other direction let M be an A -representation of $\pi_1(X, \bar{x})$. The group $\text{Aut}_A(M)$ is finite hence there is a finite étale Galois cover $\pi : P = P_\alpha \rightarrow X$ with Galois group G such that $\pi_1(X, \bar{x}) \rightarrow \text{Aut}_A(M)$ factors over G . We consider the constant sheaf M_P as a sheaf on (P, G) (see 7.1.4) by defining the map $M_P(\sigma) : \sigma_* M_P \rightarrow M_P$ ($\sigma \in G$) via

$$M_P(V \times_{P, \sigma} P) = M \xrightarrow{\sigma^{-1}} M = M_P(V) \quad (7.4)$$

where $V \rightarrow P$ is étale with V connected. Then set

$$\mathcal{F}_M := \pi_*(M_P)^G.$$

We claim that \mathcal{F}_M is étale locally isomorphic to the constant sheaf M and that this construction is independent of the choice of $\pi : P \rightarrow X$. Then \mathcal{F}_M is a *lisse* A -sheaf and the assignment $M \mapsto \mathcal{F}_M$ is functorial. To prove the claim let $V \rightarrow P$ be an étale map and assume V is connected. We have $V \times_X P \cong V \times G$. For $\sigma \in G$, the automorphism $\text{id}_V \times \sigma$ ($\sigma \in G$) on $V \times_X P$

translates via this isomorphism into the automorphism $\text{id} \times \sigma \cdot$ of $V \times G$. Hence

$$\begin{aligned}\mathcal{F}_M(V) &= \{a \in \pi_*(M_P)(V) \mid M_P(\sigma)(a) = a \text{ for all } \sigma \in G\} \\ &= \{(a_\tau) \in M \times G \mid \sigma^{-1} \cdot a_{\sigma \cdot \tau} = a_\tau \text{ for all } \sigma, \tau \in G\} \cong M,\end{aligned}$$

where the last isomorphism is given by $M \ni m \mapsto (\sigma \cdot m)_\sigma \in M \times G$. Thus $\mathcal{F}_{M|P} = M_P$. Moreover if we choose another $P' \rightarrow X$ we obtain a sheaf \mathcal{F}'_M in the same way. Then we find $P'' \rightarrow X$ dominating P' and P and we construct \mathcal{F}''_M . There is a natural map $\mathcal{F}_M \rightarrow \mathcal{F}''_M$ induced by $\pi_* M_P \rightarrow \pi''_*(M_{P|P''}) = \pi''_* M_{P''}$ and by the above we obtain

$$\mathcal{F}_{M|P''} = M_{P''} = \mathcal{F}''_{M|P''}.$$

Thus the natural map $\mathcal{F}_M \rightarrow \mathcal{F}''_M$ is an isomorphism étale locally, hence is an isomorphism globally. Same with \mathcal{F}'_M . This shows that \mathcal{F}_M is independent of the choice of $P \rightarrow X$.

We defined two functors $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ and $M \mapsto \mathcal{F}_M$ and we have to show that they are inverse to each other. It is straightforward to check $\mathcal{F}_{M, \bar{x}} = M$ (as $\pi_1(X, \bar{x})$ -representations). Now let \mathcal{F} be a lisse A -sheaf and $\mathcal{F}_{\bar{x}} = M$ its associated representation. By Proposition 7.6 there exists a finite étale Galois cover $\pi : P \rightarrow X$ with Galois group G such that $\pi^* \mathcal{F} \cong M_P$. (In particular the representation M of $\pi_1(X, \bar{x})$ factors over G .) Then

$$\mathcal{F}_M = \pi_*(\pi^* \mathcal{F})^G = \mathcal{F}.$$

Here the first equality holds by definition; for the second, first observe that there is a canonical map $\mathcal{F} \rightarrow \pi_*(\pi^* \mathcal{F})^G$ (induced by adjunction); it suffices to check that it is an isomorphism when restricted to P which follows from $\mathcal{F}_{M|P} \cong M_P$. This finishes the proof in the case A is finite.

Second case: $A = R$ is a DVR finite over \mathbb{Z}_ℓ . Let \mathfrak{m} be the maximal ideal of R . A lisse R -sheaf is a projective system $\mathcal{F} = (\mathcal{F}_n)$ as in Definition 7.9. By the first case each \mathcal{F}_n gives rise to a $\pi_1(X, \bar{x})$ -representation $\mathcal{F}_{n, \bar{x}}$ which factors over some finite quotient. Clearly they fit together to give a continuous $\pi_1(X, \bar{x})$ -representation on $\mathcal{F}_{\bar{x}} = \varprojlim_n \mathcal{F}_{n, \bar{x}}$. We already remarked in 7.1.10 that $\mathcal{F}_{\bar{x}}$ is actually a finitely generated R -module. This gives the functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ from the statement. To construct a functor in the other direction, let M be a R -representation of $\pi_1(X, \bar{x})$. We obtain induced representations on $M_n := M \otimes_R R/\mathfrak{m}^n$ which by the first case correspond to lisse R/\mathfrak{m}^n -modules \mathcal{F}_{M_n} . It is straightforward to check that we obtain in this way a projective system $\mathcal{F}_M = (\mathcal{F}_{M_n})$ which defines a lisse R -sheaf and that the functor $M \mapsto \mathcal{F}_M$ thus defined is inverse to $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$.

Third case: A is a finite field extension of \mathbb{Q}_ℓ or equal to $\bar{\mathbb{Q}}_\ell$. We get the functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ in an analogous way as above. For the functor in the other direction let V be an A -representation of $\pi_1(X, \bar{x})$. Then by Lemma 4.66 we find a DVR R finite over \mathbb{Z}_ℓ and contained in A and a finitely generated R -submodule M of V which has a continuous $\pi_1(X, \bar{x})$ -action and satisfies $M \otimes_R A \cong V$ as $A[\pi_1(X, \bar{x})]$ -modules. By the second case above M gives rise to a lisse R -sheaf \mathcal{F}_M and we define $\mathcal{F}_V := \mathcal{F}_M \otimes_R A$. One checks that this construction is independent of the choice of M and defines a functor $M \mapsto \mathcal{F}_M$ which is inverse to the functor from the statement of the theorem. This finishes the proof. \square

Example 7.14. Let k be a finite field with $q = p^n$ elements and ℓ a prime different from p . Then the lisse $\bar{\mathbb{Q}}_\ell$ -sheaves of rank 1 on $\mathrm{Spec} k$ correspond to homomorphisms $\mathrm{Gal}(\bar{k}/k) \rightarrow \bar{\mathbb{Q}}_\ell^\times$ which factor over a continuous homomorphism $\chi : \mathrm{Gal}(\bar{k}/k) \rightarrow E^\times$, with E a finite extension of \mathbb{Q}_ℓ . Let us see what this is. We have an isomorphism of topological groups $\widehat{\mathbb{Z}} \xrightarrow{\cong} \mathrm{Gal}(\bar{k}/k)$ under which 1 is mapped to the q -power Frobenius $x \mapsto x^q$. Let R be the ring of integers of E with maximal ideal $\mathfrak{m} = (\pi)$. Then a homomorphism χ as above factors over a lattice $\frac{1}{\pi^n}R$, i.e. is induced by a continuous group homomorphism $\chi_R : \widehat{\mathbb{Z}} \rightarrow \mathrm{Aut}_R(\frac{1}{\pi^n}R) = R^\times$, see Lemma 4.66. The topological group $\widehat{\mathbb{Z}}$ is the free profinite group on 1 generator, so giving a continuous map from $\widehat{\mathbb{Z}}$ to a profinite group G is the same as giving an element of G . Here R^\times is profinite and hence a homomorphism χ_R as above corresponds to the choice of an element of R^\times .

All together we see that (up to isomorphism) a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on $\mathrm{Spec} k$ of rank 1 corresponds uniquely to an element of $\bar{\mathbb{Z}}_\ell^\times$, the integral closure of \mathbb{Z}_ℓ in $\bar{\mathbb{Q}}_\ell$.

Example 7.15. Let X be a connected scheme and $\bar{x} \rightarrow X$ a geometric point. We have the lisse sheaf $\mathbb{Z}_\ell(1)$ on X (see 7.1.9) at our disposal. The corresponding representation of $\pi_1(X, \bar{x})$ is the following: In the algebraically closed field $k(\bar{x})$ take a family of elements $\zeta_n \in k(\bar{x})$, $n \geq 1$, such that ζ_n is an ℓ^n -th primitive root of unity and $\zeta_{n+1}^\ell = \zeta_n$ for all n . Let $P \rightarrow X$ be a finite étale Galois cover with Galois group G_P and a map $\bar{x} \rightarrow P$ inducing $\bar{x} \rightarrow X$. Then G_P induces an action on $k(x_P)$, where x_P is the image of $\bar{x} \rightarrow P$. If $\zeta_n \in k(x_P)$ then so are all ℓ^i -th roots of unity for $i \leq n$, and for $\sigma \in G_P$ the element $\sigma(\zeta_n)$ is again an ℓ^n -th primitive root of unity. Hence

$$\sigma(\zeta_n) = \zeta_n^{r_{n,\sigma}}, \quad \text{for some } r_{n,\sigma} \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times.$$

Let n_P be the maximal n such that $\zeta_n \in k(x_P)$. We obtain a group homomorphism

$$G_P \rightarrow (\mathbb{Z}/\ell^{n_P}\mathbb{Z})^\times, \quad \sigma \mapsto r_{n_P,\sigma}.$$

This map is independent of the choice of the ζ_n . If $P' \rightarrow X$ is another such Galois cover which factors via $P' \rightarrow P$ then we obtain a commutative diagram

$$\begin{array}{ccc} G_{P'} & \longrightarrow & (\mathbb{Z}/\ell^{n_{P'}}\mathbb{Z})^\times \\ \downarrow & & \downarrow \\ G_P & \longrightarrow & (\mathbb{Z}/\ell^{n_P}\mathbb{Z})^\times. \end{array}$$

Therefore we can take the inverse limit over an inverse system of finite étale Galois covers (P_α) pro-representing the fiber functor $F_{\bar{x}}$ and obtain a \mathbb{Z}_ℓ -representation

$$\pi_1(X, \bar{x}) \rightarrow \mathbb{Z}_\ell^\times.$$

It is the representation corresponding to the lisse sheaf $\mathbb{Z}_\ell(1)$.

Example 7.16. We can use Theorem 7.13 to construct new lisse sheaves which are related to Example 3.33. For example let k be a perfect field of characteristic $p > 1$ containing the field with $q = p^n$ elements \mathbb{F}_q and $m \geq 1$ a natural number. Then the natural inclusion $k[x] \subseteq k[x][t]/(t^q - t - x^m) := B$

defines an étale Galois covering $\pi : X := \operatorname{Spec} B \rightarrow \mathbb{A}_k^1 = \operatorname{Spec} k[x]$. (This map is étale by Definition 5.1, (e), since $\partial(t^q - t - x^m)/\partial t = -1$.) The Galois group is isomorphic to \mathbb{F}_q , where $a \in \mathbb{F}_q$ acts via $\bar{t} \mapsto \bar{t} + a$. Let $\bar{\eta} \rightarrow X$ be a geometric point over the generic point of X , we obtain

$$\mathbb{F}_q = \operatorname{coker}(\pi_1(X, \bar{\eta}) \hookrightarrow \pi_1(\mathbb{A}_k^1, \bar{\eta})).$$

Now let ℓ be a prime different from p and A an ℓ -adic coefficient ring and take a group homomorphism

$$\psi : (\mathbb{F}_q, +) \rightarrow A^\times.$$

Then the composition

$$\pi_1(\mathbb{A}_k^1, \bar{\eta}) \twoheadrightarrow \mathbb{F}_q \xrightarrow{\psi} A^\times = \operatorname{Aut}_A(A)$$

is an A -representation of $\pi_1(\mathbb{A}_k^1, \bar{\eta})$ of rank 1. Hence it defines a free lisse A -sheaf $\mathcal{L}_{m, \psi}$ of rank 1 on \mathbb{A}_k^1 . Its sheaf theoretic description is as follows: If A is finite, then denote by L_ψ the sheaf on (X, \mathbb{F}_q) (see 7.1.4) whose underlying sheaf on X is the constant sheaf A and where the action of $a \in \mathbb{F}_q$ is given by multiplication with $\psi(a)$. Concretely, if $\bar{x} \rightarrow X$ is a geometric point we get an isomorphism

$$L_{\psi, \bar{x}} = A \rightarrow A = L_{\psi, \bar{x}+a}, \quad b \mapsto \psi(a)b,$$

where $\bar{x} + a$ is the composition $\bar{x} \rightarrow X \xrightarrow{+a} X$. Then $\mathcal{L}_\psi = \pi_*(L_\psi)^{\mathbb{F}_q}$. One obtains the description for general (infinite) A by the usual limit procedure.

Example 7.17. One can generalize the above example for the case $m = 1$ as follows (see e.g. [12, Sommes trig. 1.]): Let \mathbb{F}_q be the finite field with $q = p^n$ elements and G a commutative algebraic group scheme over \mathbb{F}_q . Let $F : G \rightarrow G$ be the Frobenius morphism (it is given by $\mathcal{O}_G \rightarrow \mathcal{O}_G, x \mapsto x^p$). Then $F^n = (F \circ \dots \circ F)$ (n -times) is an \mathbb{F}_q -endomorphism of G . We obtain a finite étale Galois covering (the *Lang isogeny*)

$$F - \operatorname{id}_G : G \rightarrow G$$

whose kernel and Galois group is the finite group $G(\mathbb{F}_q)$. Let A be an ℓ -adic coefficient ring and $\chi : G(\mathbb{F}_q) \rightarrow A^\times$ a homomorphism. Then the composition

$$\pi_1(G, \bar{\eta}) \twoheadrightarrow \operatorname{coker}(\pi_1(G, \bar{\eta}) \xrightarrow{F - \operatorname{id}} \pi_1(G, \bar{\eta})) = G(\mathbb{F}_q) \xrightarrow{\chi} A^\times$$

is an A -representation of $\pi_1(G, \bar{\eta})$. Hence we obtain a lisse rank 1 sheaf \mathcal{L}_χ on G . In case $G = \mathbb{A}_{\mathbb{F}_q}^1$ we obtain the example above for $m = 1$.

8. ℓ -ADIC COHOMOLOGY

In this section we recall without proofs the main properties of étale cohomology on a scheme over an algebraically closed field with coefficients in a lisse ℓ -adic sheaf. The references for this section are [5], [12] and [19].

Convention 8.1. Let k be a perfect field of characteristic $p \geq 0$ and ℓ a prime number which is invertible in k . Throughout this section a k -scheme is a scheme which is *separated* and of finite type over k .

8.1.1. *Étale cohomology.* Let $f : X \rightarrow Y$ be a morphism of k -schemes.

- (a) Let A be a torsion ring. The category of A -modules on $X_{\text{ét}}$ has enough injectives. Therefore we can define the i -th higher direct image of an A -module \mathcal{F} on $X_{\text{ét}}$ under f as the i -th right derived functor of f_* , i.e.

$$R^i f_* \mathcal{F} := H^i(f_* \mathcal{I}^\bullet),$$

where $\mathcal{F} \rightarrow \mathcal{I}^\bullet$ is an injective resolution of A -modules on $X_{\text{ét}}$.

Choose a compactification of f (using Nagata's theorem [11], [34]), i.e. a proper morphism $\bar{f} : \bar{X} \rightarrow Y$ together with a dominant open immersion $j : X \hookrightarrow \bar{X}$ such that $f = \bar{f} \circ j$. Then we define

$$R^i f_! \mathcal{F} := R^i \bar{f}_* j_! \mathcal{F}.$$

This definition is independent of the choice of the compactification, see [12, Arcata, IV, (5.3)].

In case k is *algebraically closed* and $\pi : X \rightarrow \text{Spec } k$ is the structure map we also write

$$H^i(X, \mathcal{F}) := R^i \pi_* \mathcal{F} \quad \text{and} \quad H_c^i(X, \mathcal{F}) := H^i(\bar{X}, j_! \mathcal{F})$$

and call it the i -th cohomology and the i -th *cohomology with compact support* of \mathcal{F} , respectively. (The A -module $H^i(X, \mathcal{F})$ is in this case also equal to the i -th right derived functor of the global section functor $\Gamma(X, -)$.)

- (b) Assume k is algebraically closed. Let A be an ℓ -adic coefficient ring (in the sense of Convention 7.12) and let \mathcal{F} be a constructible A -sheaf on $X_{\text{ét}}$. Then we find a DVR R which is finite over \mathbb{Z}_ℓ such that A is an R -algebra, and a constructible R -sheaf $(\mathcal{F}_n)_n$ such that $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$. We define the i -th étale cohomology group of \mathcal{F} , respectively the i -th étale cohomology group with compact support of \mathcal{F} as

$$H^i(X, \mathcal{F}) := (\varprojlim_n H^i(X, \mathcal{F}_n)) \otimes_R A, \quad H_c^i(X, \mathcal{F}) = (\varprojlim_n H_c^i(X, \mathcal{F}_n)) \otimes_R A.$$

This definition is independent of the choice of R and the constructible R -sheaf (\mathcal{F}_n) , see [19, VI, 2.2].

Remark 8.2. Let \bar{k} be an algebraic closure of k , A a finite ℓ -adic coefficient ring, $\pi : X \rightarrow \text{Spec } k$ a k -scheme and \mathcal{F} a constructible A -sheaf. Then $R^i \pi_* \mathcal{F}$ and $R^i \pi_! \mathcal{F}$ are lisse A -sheaves on $\text{Spec } k$ and the corresponding $\text{Gal}(\bar{k}/k)$ representations are equal to $H^i(X \otimes_k \bar{k}, \mathcal{F})$ and $H_c^i(X \otimes_k \bar{k}, \mathcal{F})$ respectively. (This follows from the smooth base change theorem for $R^i \pi_*$ ([5, XVI, Cor 1.2]) and the base change theorem for $R^i \pi_!$ ([5, XVII, Prop. 5.2.8]).)

8.1.2. In the following we assume that k is *algebraically closed*. Let $f : X \rightarrow Y$ be a morphism of k -schemes, A an ℓ -adic coefficient ring and \mathcal{F} a constructible A -sheaf on $X_{\text{ét}}$. We list some properties:

- (a) If A is finite (i.e. $A = R/\mathfrak{m}^n$ for some DVR R finite over \mathbb{Z}_ℓ), then $R^i f_! \mathcal{F}$ and $R^i f_* \mathcal{F}$ are constructible on Y , see [12, Arcata, IV, Thm (6.2)] and [12, Th. finitude, Thm 1.1]. If A is general then $H^i(X, \mathcal{F})$ and $H_c^i(X, \mathcal{F})$ are finitely generated A -modules, cf. [19, VI, 2.2].

- (b) Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be a short exact sequence of constructible A -sheaves. Then there is a long exact sequence

$$\cdots \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(X, \mathcal{F}'') \rightarrow H^{i+1}(X, \mathcal{F}') \rightarrow \cdots,$$

and similarly with H^i replaced by H_c^i . (If A is finite this follows immediatly from the general theory of right derived functors and the fact that $j_!$ is exact. For general A notice that the groups $H^i(X, \mathcal{F}_n)$ are finite by (a); hence any projective system of subquotients of $(H^i(X, \mathcal{F}_n))$ satisfies the Mittag-Leffler condition; hence the long exact sequence of pro-groups gives a long exact sequence in the limit; further if A is not finite, then it is flat over R and hence tensoring with A over R is exact.)

- (c) Assume $f : X \rightarrow Y$ is finite. If A is finite then $R^i f_* \mathcal{F} = 0$ for all $i \geq 1$, see [12, Arcata, II, Prop. (3.6)]. This implies (via a Leray spectral sequence argument) that for general A we have $H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})$.
- (d) Assume X is a proper k -scheme and $j : U \hookrightarrow X$ is an open subscheme with complement $i : Z \hookrightarrow X$. Then there is a long exact sequence

$$\cdots \rightarrow H_c^i(U, j^* \mathcal{F}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(Z, i^* \mathcal{F}) \rightarrow H_c^{i+1}(U, j^* \mathcal{F}) \rightarrow \cdots.$$

If we write $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$ then this sequence is induced as above by the exact sequences $0 \rightarrow j_! j^* \mathcal{F}_n \rightarrow \mathcal{F}_n \rightarrow i_* i^* \mathcal{F}_n \rightarrow 0$, for $n \geq 1$.

- (e) Assume X is affine. Then $H^i(X, \mathcal{F}) = 0$ for all $i > \dim X$, see [5, XIV, Cor. 3.2].
- (f) We have $H^i(X, \mathcal{F}) = 0$ for all $i > 2 \dim X$, see [5, X, Cor. 4.3].
- (g) Assume $k = \mathbb{C}$ and X is smooth over \mathbb{C} . Then

$$H^i(X, \mathbb{Z}/n\mathbb{Z}) \cong H^i(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \quad \text{and} \quad H_c^i(X, \mathbb{Z}/n\mathbb{Z}) \cong H_c^i(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}),$$

where $n \geq 1$ and the right-hand side of the two isomorphisms is the singular cohomology (resp. with compact supports) of the complex manifold $X(\mathbb{C})$. See [5, XI, Thm. 4.4] and [12, Arcata, IV, Thm (6.3)]

- (h) *Cohomology with support.* Let $j : U \hookrightarrow X$ be an open subset of X with complement $i : Z \hookrightarrow X$. If A is a finite ℓ -adic coefficient ring and \mathcal{F} a constructible A -sheaf on X we define

$$i^! \mathcal{F} := \text{Ker}(i^* \mathcal{F} \rightarrow i^* j_* j^* \mathcal{F}).$$

We obtain a left exact functor $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F}) := \Gamma(Z, i^! \mathcal{F})$. Its right derived functor is denoted by $H_Z^i(X, \mathcal{F})$.

If A is a general ℓ -adic coefficient ring, we write $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$ and set

$$H_Z^i(X, \mathcal{F}) := (\varprojlim_n H_Z^i(X, \mathcal{F}_n)) \otimes_R A.$$

We have an exact sequence

$$\cdots \rightarrow H^i(X, \mathcal{F}) \rightarrow H^i(U, \mathcal{F}|_U) \rightarrow H_Z^{i+1}(X, \mathcal{F}) \rightarrow \cdots.$$

(In case A is finite this is the usual localization sequence. For general A it follows from this and (a) that $H_Z^i(X, \mathcal{F}_n)$ is finite and hence as above taking the inverse limit over the sequences for \mathcal{F}_n and tensoring with A gives the exact sequence in general.)

Remark 8.3. Let X be a smooth \mathbb{C} -scheme. Notice that 8.1.2, (g) also implies

$$H^i(X, \mathbb{Z}_\ell) \cong H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell, \quad \text{for all primes } \ell,$$

where \mathbb{Z}_ℓ on the left-hand side denotes the constant lisse \mathbb{Z}_ℓ -sheaf $(\mathbb{Z}/\ell^n)_n$. Indeed, $H^i(X(\mathbb{C}), \mathbb{Z})$ is a finitely generated \mathbb{Z} -module, hence

$$H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \varprojlim_n (H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n).$$

On the other hand the exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot \ell^n} \mathbb{Z} \rightarrow \mathbb{Z}/\ell^n \rightarrow 0$ yields an exact sequence

$$0 \rightarrow H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n \rightarrow H^i(X(\mathbb{C}), \mathbb{Z}/\ell^n) \rightarrow {}_{\ell^n}H^{i+1}(X(\mathbb{C}), \mathbb{Z}) \rightarrow 0,$$

where we denote by ${}_{\ell^n}(-)$ the kernel of the multiplication by ℓ^n . Using (g) and taking \varprojlim_n we arrive at a short exact sequence

$$0 \rightarrow H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow H^i(X, \mathbb{Z}_\ell) \rightarrow N \rightarrow 0,$$

where N consists of the sequences (a_n) with $a_n \in H^{i+1}(X(\mathbb{C}), \mathbb{Z})$, $\ell^n \cdot a_n = 0$, and such that $a_{n-1} = \ell \cdot a_n$. In particular all the a_n are ℓ -power-torsion. Thus there exists an $n_0 \geq 1$ such that $\ell^{n_0} \cdot a_n = 0$ for all n (since $H^{i+1}(X(\mathbb{C}), \mathbb{Z})$ is finitely generated over \mathbb{Z}). Hence $a_n = \ell^{n_0} \cdot a_{n+n_0} = 0$, i.e. $N = 0$.

Theorem 8.4 (Poincaré duality). *Let X be a smooth k -scheme of pure dimension d and \bar{k} an algebraic closure of k . Let A be an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf. Then for all $i \in \mathbb{Z}$ there is a natural (i.e. functorial in \mathcal{F}) and $\text{Gal}(\bar{k}/k)$ -equivariant isomorphism*

$$H^{2d-i}(X \otimes_k \bar{k}, \mathcal{F}^\vee(d)) \xrightarrow{\sim} H_c^i(X \otimes_k \bar{k}, \mathcal{F})^\vee.$$

Here (d) is the Tate twist 7.1.9, $H_c^i(X \otimes_k \bar{k}, \mathcal{F})^\vee := \text{Hom}_A(H_c^i(X \otimes_k \bar{k}, \mathcal{F}), A)$ and \mathcal{F}^\vee is defined as follows: write $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$ with a lisse R -sheaf (\mathcal{F}_n) , then $\mathcal{F}^\vee := (\mathcal{H}om(\mathcal{F}_n, R/\mathfrak{m}^n)) \otimes_R A$ is a lisse A -sheaf.

Proof. If A is finite this is [5, XVIII, (3.2.6.2)]. (In *loc. cit.* it is done for $A = \mathbb{Z}/\ell^n$, but since for any DVR R with maximal ideal \mathfrak{m} , the ring R/\mathfrak{m}^n is self injective, the same argument works.) In general we can assume that A is flat over R and write $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$ with (\mathcal{F}_n) a lisse R -sheaf. Since $\mathcal{F}_n = \mathcal{F}_{n+1} \otimes R/\mathfrak{m}^n$ we have $\varprojlim_m \mathcal{H}om(\mathcal{F}_m, R/\mathfrak{m}^n) = \mathcal{H}om(\mathcal{F}_n, R/\mathfrak{m}^n)$. Therefore applying $\varprojlim_n \varinjlim_m$ to the Poincaré isomorphism for the \mathcal{F}_n and tensoring over R with A we obtain a $\text{Gal}(\bar{k}/k)$ -equivariant isomorphism

$$H^{2d-i}(X \otimes_k \bar{k}, \mathcal{F}^\vee(d)) \xrightarrow{\sim} (\varprojlim_n \varinjlim_m \text{Hom}_R(H_c^i(X \otimes_k \bar{k}, \mathcal{F}_m), R/\mathfrak{m}^n)) \otimes_R A.$$

Set $H^i := \varprojlim_m H_c^i(X \otimes_k \bar{k}, \mathcal{F}_m)$. It is a finitely generated R -module. By [19, VI, Lem 2.2.2] $H^i = \varprojlim_m (H^i \otimes_R R/\mathfrak{m}^m)$. We obtain

$$\begin{aligned} \varprojlim_n \varinjlim_m \text{Hom}_R(H_c^i(X \otimes_k \bar{k}, \mathcal{F}_m), R/\mathfrak{m}^n) &= \varprojlim_n \text{Hom}_R(H^i, R/\mathfrak{m}^n) \\ &= \text{Hom}_R(H^i, R). \end{aligned}$$

Since R is noetherian, H^i is actually a finitely presented R -module and since A is flat over R we obtain $\text{Hom}_R(H^i, R) \otimes_R A \cong \text{Hom}_R(H^i \otimes_R A, A)$. Putting all the isomorphisms together we obtain the statement. \square

Corollary 8.5. *Let X be a smooth affine k -scheme of pure dimension d , A an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on X . Then*

$$H_c^i(X \otimes_k \bar{k}, \mathcal{F}) = 0, \quad \text{for all } i < d.$$

Proof. This follows from Poincaré duality and 8.1.2, (e). \square

Theorem 8.6 (Purity). *Let $i : Y \hookrightarrow X$ be a closed immersion of pure codimension c between smooth k -schemes, A an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf. Then there is a natural $\text{Gal}(\bar{k}/k)$ -equivariant isomorphism*

$$H^j(Y \otimes_k \bar{k}, i^* \mathcal{F}) \xrightarrow{\sim} H_{Y \otimes_k \bar{k}}^{j+2c}(X \otimes_k \bar{k}, \mathcal{F}(c)), \quad \text{for all } j,$$

where (c) on the right-hand side denotes the Tate twist. In particular $H_{Y \otimes_k \bar{k}}^j(X \otimes_k \bar{k}, \mathcal{F}) = 0$ for all $j < 2c$.

Proof. If A is finite we have $R^j i^! \mathcal{F} = 0$ and $R^{2c} i^! \mathcal{F} = i^* \mathcal{F}(-c)$, by [5, XVI, Cor 3.8, Rem. 3.10, a)]. Therefore the statement follows in this case from the local-global spectral sequence

$$E_2^{a,b} = H^a(X \otimes_k \bar{k}, i_* R^b i^! \mathcal{F}(c)) \Rightarrow H_{Y \otimes_k \bar{k}}^*(X \otimes_k \bar{k}, \mathcal{F}(c)).$$

For general A applying \varprojlim_n and $\otimes_R A$ gives the assertion. \square

Theorem 8.7 (Lefschetz trace formula, [12, Cycle, Cor 3.7]). *Let X be a smooth projective scheme over an algebraically closed field k of characteristic $p > 0$ and $\ell \neq p$ a prime number. Let $f : X \rightarrow X$ be a k -morphism and assume that the graph Γ_f of f and the diagonal Δ_X intersect properly in $X \times X$ (i.e. the intersection scheme $\Gamma_f \cap \Delta_X$ is either empty or zero dimensional). Then*

$$(\Gamma_f \cdot \Delta_X) = \sum_i (-1)^i \text{Tr}(f^* | H^i(X, \mathbb{Q}_\ell)).$$

Here the left-hand side is the degree of the intersection product $\Gamma_f \cdot \Delta_X$, concretely

$$(\Gamma_f \cdot \Delta_X) := \sum_{x \in \Gamma_f \cap \Delta_X} \text{length}(\mathcal{O}_{X,x}/I_f), \quad (8.1)$$

where $I_f \subseteq \mathcal{O}_{X,x}$ is the ideal generated by the elements $a - f^*(a)$, $a \in \mathcal{O}_{X,x}$.

Lemma 8.8. *Let k be an algebraically closed field, X an integral k -scheme with generic point η , E a finite extension of \mathbb{Q}_ℓ with ring of integers R and \mathcal{F} a constructible E -sheaf on X . Suppose there is a finite group G acting on \mathcal{F} , i.e. there is a constructible R -sheaf $\mathcal{F}' = (\mathcal{F}_n)$ such that the \mathcal{F}_n form a projective system of $R[G]$ -modules and $\mathcal{F} = \mathcal{F}' \otimes_R E$. Then the G -invariant sections \mathcal{F}_n^G form a projective system which induces a constructible R -sheaf denoted by $(\mathcal{F}')^G$. We define the constructible E -sheaf*

$$\mathcal{F}^G := (\mathcal{F}')^G \otimes_R E.$$

Then

$$H^i(X, \mathcal{F}^G) = H^i(X, \mathcal{F})^G.$$

Proof. Set $N := |G|$. For all $n_1 \geq n$ we have $\mathcal{F}_{n_1} \otimes_R R/\mathfrak{m}^n = \mathcal{F}_n$ by definition. We obtain natural maps

$$(\mathcal{F}_{n_1})^G \otimes_R R/\mathfrak{m}^n \rightarrow \mathcal{F}_n^G, \quad (8.2)$$

the cokernel of which is killed by N . (Indeed, it suffices to check this on the stalks. If we take a local section $s_n \in \mathcal{F}_n^G$ we can lift it to a local section $s \in \mathcal{F}_{n_1}$. Then $s' := \sum_{\sigma \in G} \sigma(s)$ is a G -invariant section of $\mathcal{F}_{n_1}^G$ which modulo \mathfrak{m}^n equals $N \cdot s_n$.) Now the stalks of \mathcal{F}_n^G are finite (as sets). It follows that the images of the maps (8.2) become stationary for $n_1 \rightarrow \infty$. We define $(\mathcal{F}')_n^G \subseteq \mathcal{F}_n^G$ to be the intersection over the images of all these maps. Then $(\mathcal{F}')_n^G$ is a constructible R/\mathfrak{m}^n -subsheaf of \mathcal{F}_n^G and by construction $(\mathcal{F}')_{n+1}^G \otimes_R R/\mathfrak{m}^n = (\mathcal{F}')_n^G$. We obtain the constructible R -sheaf $(\mathcal{F}')^G := ((\mathcal{F}')_n^G)$ from the first part of the Lemma. Since the cokernel of the inclusion $(\mathcal{F}')_n^G \hookrightarrow \mathcal{F}_n^G$ is killed by N we obtain:

$$H^i(X, \mathcal{F}^G) \stackrel{\text{defn}}{=} \varprojlim_n H^i(X, (\mathcal{F}')_n^G) \otimes_R E = \varprojlim_n H^i(X, \mathcal{F}_n^G) \otimes_R E.$$

Let $\mathcal{F}_n \rightarrow \mathcal{I}^\bullet$ be a resolution of \mathcal{F}_n by injective $R/\mathfrak{m}^n[G]$ -modules on $X_{\text{ét}}$. The sheaf of G -invariants $(\mathcal{I}^j)^G$ is an injective sheaf of R/\mathfrak{m}^n -modules for all j . (Since the functor of taking G -invariants from the category of $R/\mathfrak{m}^n[G]$ -modules on $X_{\text{ét}}$ to the category of R/\mathfrak{m}^n -modules on $X_{\text{ét}}$ has an exact left adjoint, namely the functor which to an R/\mathfrak{m}^n -module M associates the $R/\mathfrak{m}^n[G]$ -module given by the same M with trivial G -action.) Further $H^0(\mathcal{I}^{\bullet G}) = \mathcal{F}_n^G$ and $H^j(\mathcal{I}^{\bullet G})$ is N -torsion for all $j \neq 0$, by a similar argument as above. It follows that kernel and cokernel of $H^i(X, \mathcal{F}_n^G) \rightarrow H^i(\Gamma(X, \mathcal{I}^{\bullet G}))$ are killed by N . On the other hand $H^i(\Gamma(X, \mathcal{I}^{\bullet G})) = H^i(\Gamma(X, \mathcal{I}^\bullet)^G)$ and kernel and cokernel of the natural map $H^i(\Gamma(X, \mathcal{I}^\bullet)^G) \rightarrow H^i(\Gamma(X, \mathcal{I}^\bullet))^G = H^i(X, \mathcal{F}_n)^G$ are killed by N . All together we obtain a natural map

$$H^i(X, \mathcal{F}_n^G) \rightarrow H^i(X, \mathcal{F}_n)^G$$

whose kernel and cokernel are killed by $2N$. Taking the limit and tensoring with E we arrive at the statement. \square

9. GROTHENDIECK-OGG-SHAFAREVICH

In this section we explain Grothendieck's proof of the Grothendieck-Ogg-Shafarevich formula, partly following Katz. References for this section are [19, Exp. X], [25, Ch. 2], see also [41].

9.0.1. Throughout this section we fix the following notation:

- k is a perfect field of characteristic $p > 0$ and \bar{k} an algebraic closure.
- ℓ is a prime number different from p .
- C is a smooth proper and geometrically connected curve over k .
- $U \subseteq C$ is a *strict* open subset (in particular it is affine).
- $K = k(C)$ is the function field of C , \bar{K} an algebraic closure and $K^{\text{sep}} \subseteq \bar{K}$ a separable closure. We denote by $\eta : \text{Spec } K \rightarrow C$ the generic point of C and by $\bar{\eta} : \text{Spec } \bar{K} \rightarrow C$ the induced geometric point over η .
- For a closed point $x \in C$ we denote by K_x the completion of K with respect to the valuation $v_x : K^\times \rightarrow \mathbb{Z}$ corresponding to x and by K_x^{sep} a separable closure. For all x we choose an embedding $\iota_x : K^{\text{sep}} \hookrightarrow K_x^{\text{sep}}$ over K .

- $G = \text{Gal}(K^{\text{sep}}/K)$ is the absolute Galois group of K and for a closed point $x \in C$ we denote by $D_x := D_x^{\iota_x}$ the image of the inclusion $\text{Gal}(K_x^{\text{sep}}/K_x) \hookrightarrow G$ induced by ι_x , i.e. D_x is the decomposition subgroup of G with respect to ι_x . We denote by $P_x \subseteq I_x \subseteq D_x$ the wild inertia and inertia subgroups, respectively. (Notice that if we take a different K -embedding $\iota_{x,1} : K^{\text{sep}} \hookrightarrow K_x^{\text{sep}}$, then there exists a $\sigma \in G$ such that $\iota_{x,1} = \iota_x \circ \sigma$; thus the resulting decomposition group is conjugate to D_x , i.e. $D_x^{\iota_{x,1}} = \sigma^{-1} D_x \sigma$. Same with I_x, P_x . Hence D_x, P_x, I_x are only well-defined up to conjugation by elements in G .)

9.1.1. Let A be an ℓ -adic coefficient ring (Convention 7.12) and \mathcal{F} a free lisse A -module on U . By Theorem 7.13, we can identify \mathcal{F} with a representation of $\pi_1(U, \bar{\eta})$ on $\mathcal{F}_{\bar{\eta}}$. The natural surjection $G \rightarrow \pi_1(U, \bar{\eta})$ hence induces a G -action on $\mathcal{F}_{\bar{\eta}}$, which we can further restrict to a P_x -action, $x \in C$. The action of P_x on $\mathcal{F}_{\bar{\eta}}$ factors over a finite quotient of P_x , see Lemma 4.69. Thus the Swan conductor of the P_x -representation on $\mathcal{F}_{\bar{\eta}}$ is defined (Definition 4.81) and denoted by

$$\text{Swan}_x(\mathcal{F}).$$

We observe that $\text{Swan}_x(\mathcal{F})$ is independent of the chosen embedding $\iota_x : K^{\text{sep}} \hookrightarrow K_x^{\text{sep}}$. Indeed, if we choose a different embedding there is a $\sigma \in G$ such that we have to consider the restriction of the G -action on $\mathcal{F}_{\bar{\eta}}$ to the group $\sigma^{-1} P_x \sigma$. But then σ maps the break decomposition of $\mathcal{F}_{\bar{\eta}}$ (Definition 4.79) with respect to $\sigma^{-1} P_x \sigma$ isomorphically to the break decomposition of $\mathcal{F}_{\bar{\eta}}$ with respect to P_x . Hence we obtain the same number $\text{Swan}_x(\mathcal{F})$, when we compute it with respect to the $\sigma^{-1} P_x \sigma$ -action.

We recall the following facts (see Remark 4.82, Theorem 4.85):

- (a) Let A' be an ℓ -adic coefficient ring and $A \rightarrow A'$ a ring homomorphism. Then for all $x \in C$

$$\text{Swan}_x(\mathcal{F} \otimes_A A') = \text{Swan}_x(\mathcal{F}).$$

- (b) $\text{Swan}_x(\mathcal{F}) \in \mathbb{Z}_{\geq 0}$.
(c) $\text{Swan}_x(\mathcal{F}) = 0 \iff \mathcal{F}$ is tame at x , i.e. $(\mathcal{F}_{\bar{\eta}})^{P_x} = \mathcal{F}_{\bar{\eta}}$. In particular, $\text{Swan}_x(\mathcal{F}) = 0$ for all $x \in U$.
(d) Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of free lisse A -sheaves on U . Then

$$\text{Swan}_x(\mathcal{F}) = \text{Swan}_x(\mathcal{F}') + \text{Swan}_x(\mathcal{F}'').$$

9.1.2. Recall that the compactly supported Euler characteristic of a lisse $\bar{\mathbb{Q}}_{\ell}$ -sheaf \mathcal{F} on U is defined to be

$$\begin{aligned} \chi_c(\bar{U}, \mathcal{F}) &= \sum_i (-1)^i \dim_{\bar{\mathbb{Q}}_{\ell}} H_c^i(\bar{U}, \mathcal{F}) \\ &= -\dim_{\bar{\mathbb{Q}}_{\ell}} H_c^1(\bar{U}, \mathcal{F}) + \dim_{\bar{\mathbb{Q}}_{\ell}} H_c^2(\bar{U}, \mathcal{F}), \end{aligned}$$

where $\bar{U} = U \otimes_k \bar{k}$ and we abuse notation denoting the pullback of \mathcal{F} to \bar{U} still by \mathcal{F} .

The *Grothendieck-Ogg-Shafarevich* formula is the following theorem.

Theorem 9.1. *Let \mathcal{F} be a lisse $\bar{\mathbb{Q}}_{\ell}$ -sheaf on U . Then*

$$\chi_c(\bar{U}, \mathcal{F}) = \text{rk}(\mathcal{F}) \cdot \chi_c(\bar{U}, \bar{\mathbb{Q}}_{\ell}) - \sum_{x \in C \setminus U} [k(x) : k] \cdot \text{Swan}_x(\mathcal{F}).$$

Before we prove the theorem we give some preliminary steps.

Lemma 9.2. *Let $\bar{x} \in C \otimes_k \bar{k}$ be a closed point mapping to $x \in C$. Let A be an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on U . Then*

$$\mathrm{Swan}_x(\mathcal{F}) = \mathrm{Swan}_{\bar{x}}(\mathcal{F}),$$

where here as everywhere we abuse notation and write \mathcal{F} on the right-hand side instead of $\mathcal{F}|_{U \otimes_k \bar{k}}$.

Proof. The point \bar{x} corresponds to a prime ideal in $k(x) \otimes_k \bar{k}$ which by Hensel's Lemma corresponds uniquely to a prime ideal in $K_x \otimes_k \bar{k}$. Denote by L the completion of the function field of $U \otimes_k \bar{k}$ with respect to \bar{x} ; it is the localization of $K_x \otimes_k \bar{k}$ with respect to the prime ideal corresponding to \bar{x} . Then L is a complete discrete valuation field with residue field \bar{k} . Since k is perfect, the canonical inclusion $K_x \hookrightarrow L$ is unramified. It follows that L is isomorphic to the maximal unramified extension of K_x inside K_x^{sep} , hence its absolute Galois group is isomorphic to the inertia group I_x . Hence the restriction to I_x of the representation corresponding to \mathcal{F} is isomorphic to the representation corresponding to $\mathcal{F}|_{U \otimes_k \bar{k}}$. The claim follows. \square

Lemma 9.3. *Assume $k = \bar{k}$. Let A be an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on U . Then there exists a two term complex of A -modules*

$$C(\mathcal{F}) : C^1(\mathcal{F}) \rightarrow C^2(\mathcal{F})$$

with the following properties:

- (a) $H^i(C(\mathcal{F})) = H_c^i(U, \mathcal{F})$, all i .
- (b) If \mathcal{F} is free then $C^i(\mathcal{F})$ is a free A -module of finite rank.
- (c) The functor $\mathcal{F} \mapsto C(\mathcal{F})$ is exact.
- (d) If A' is an ℓ -adic coefficient ring and $A \rightarrow A'$ a ring homomorphism and \mathcal{F} is free, we have a canonical isomorphism $C(\mathcal{F}) \otimes_A A' \cong C(\mathcal{F} \otimes_A A')$.

Proof. We are following the proof given in [25, Lem. 2.2.7]. Denote by $j : U \hookrightarrow C$ the inclusion of U into C . Pick once and for all a closed point $P \in U$. It is a closed subscheme of U of codimension 1 which is smooth over k . Thus the long exact localization sequence 8.1.2, (h) together with purity (Theorem 8.6) gives a long exact sequence

$$\dots \rightarrow H^i(C, j_! \mathcal{F}) \rightarrow H^i(C \setminus P, (j_! \mathcal{F})|_{C \setminus P}) \rightarrow H^{i-1}(P, (j_! \mathcal{F})(-1)|_P) \rightarrow \dots$$

By definition $H^i(C, j_! \mathcal{F}) = H_c^i(U, \mathcal{F})$, which vanishes for $i \neq 1, 2$ by Corollary 8.5. By 8.1.2, (e), we have

$$H^i(P, (j_! \mathcal{F})(-1)|_P) = 0, \quad \text{for } i \neq 0. \tag{9.1}$$

From this and 8.1.2, (e) we get

$$H^i(C \setminus P, (j_! \mathcal{F})|_{C \setminus P}) = 0, \quad \text{for } i \neq 1. \tag{9.2}$$

Set

$$C^1(\mathcal{F}) := H^1(C \setminus P, (j_! \mathcal{F})|_{C \setminus P}), \quad C^2(\mathcal{F}) := H^0(P, (j_! \mathcal{F})(-1)|_P).$$

By (9.1) and (9.2) the functors $\mathcal{F} \mapsto C^i(\mathcal{F})$ are exact and by the long exact sequence above we get a complex $C^1(\mathcal{F}) \rightarrow C^2(\mathcal{F})$ whose cohomology groups

are equal to $H_c^i(U, \mathcal{F})$. It remains to prove (b) and (d). Notice that both statements are obvious if A is a field extension of \mathbb{Q}_ℓ . Thus we may assume that there exists a DVR R finite over \mathbb{Z}_ℓ that surjects onto A . In this case we can represent \mathcal{F} as a projective system (\mathcal{F}_n) and by definition $C^i(\mathcal{F}) = \varprojlim_n C^i(\mathcal{F}_n)$. Since $C^i(\mathcal{F})$ is a finitely generated A -module it suffices to prove (b) for A finite, i.e. $A = R/\mathfrak{m}^n$. Furthermore let $\pi \in \mathfrak{m}$ be a local parameter of R . Then we have an exact sequence $0 \rightarrow \mathcal{F} \xrightarrow{\cdot\pi^n} \mathcal{F} \rightarrow \mathcal{F}_n \rightarrow 0$, which yields $C^i(\mathcal{F})/\pi^n \cong C^i(\mathcal{F}_n)$. If $R \rightarrow R'$ is a finite extension we get

$$C^i(\mathcal{F}) \otimes_R R' = \varprojlim_n (C^i(\mathcal{F}) \otimes_{R/\mathfrak{m}^n} R'/\mathfrak{m}'^n) = \varprojlim_n (C^i(\mathcal{F}_n) \otimes_{R/\mathfrak{m}^n} R'/\mathfrak{m}'^n).$$

Thus in (d) it suffices to consider A (and hence also A') finite.

So let A be a finite local ring with residue characteristic prime to p and \mathcal{F} a lisse A -sheaf on U . If M is a finitely generated A -module we find a resolution $P_\bullet \rightarrow M$ by free finitely generated A -modules. We can view M and the P_j as constant sheaves on U and form the complex of lisse A -sheaves $\mathcal{F} \otimes_A P_\bullet$, which is augmented towards $\mathcal{F} \otimes_A M$. The j -th homology of this complex is by definition $\text{Tor}_j^A(\mathcal{F}, M)$, which is a lisse A -sheaf. Since P_\bullet is a complex of free A -modules, we obtain $C^i(\mathcal{F} \otimes_A P_\bullet) = C^i(\mathcal{F}) \otimes_A P_\bullet$. All together the exactness of the C^i gives

$$C^i(\text{Tor}_j^A(\mathcal{F}, M)) = H_j(C^i(\mathcal{F} \otimes_A P_\bullet)) = H_j(C^i(\mathcal{F}) \otimes_A P_\bullet) = \text{Tor}_j^A(C^i(\mathcal{F}), M).$$

In case \mathcal{F} is free, the left-hand side vanishes for $j \geq 1$ thus $C^i(\mathcal{F})$ is a finitely generated flat A -module, hence is free. Furthermore, the case $j = 0$ gives the base change statement. This finishes the proof. \square

As an immediate consequence we get:

Corollary 9.4. *Assume $k = \bar{k}$. Let \mathcal{F} be a free lisse A -sheaf on U and $C(\mathcal{F})$ the complex from Lemma 9.3. Then we define*

$$\chi_c(U, \mathcal{F}) := -\text{rk}_A(C^1(\mathcal{F})) + \text{rk}_A(C^2(\mathcal{F})).$$

We have:

(a) *If $H_c^i(U, \mathcal{F})$ is a free A -module, for $i = 1, 2$, then*

$$\chi_c(U, \mathcal{F}) = \sum_i (-1)^i \text{rk}_A H_c^i(U, \mathcal{F}).$$

(b) *If A' is an ℓ -adic coefficient ring and $A \rightarrow A'$ a ring homomorphism, then*

$$\chi_c(U, \mathcal{F}) = \chi_c(U, \mathcal{F} \otimes_A A').$$

(c) *If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of free lisse A -sheaves on U , then*

$$\chi_c(U, \mathcal{F}) = \chi_c(U, \mathcal{F}') + \chi_c(U, \mathcal{F}'').$$

9.1.3. Assume $k = \bar{k}$. Let $U' \rightarrow U$ be a connected finite étale Galois covering with Galois group $G_{U'} = \text{Gal}(K'/K)$, where $K' = k(C')$. We can extend it uniquely to a finite morphism $C' \rightarrow C$ between smooth proper and connected curves over k . Let $x' \in C' \setminus U'$ be a closed point lying over $x \in C \setminus U$ and $G_{U', x'} = \{\sigma \in G_{U'} \mid \sigma(x') = x'\} = \text{Gal}(K'_{x'}/K_x)$ the decomposition group of

$G_{U'}$ at x' . By Theorem 4.53 there exists a finitely generated and projective $\mathbb{Z}_\ell[G_{U',x'}]$ -module

$$\mathrm{Sw}_{G_{U',x'}}$$

underlying the Swan representation of $G_{U',x'}$. We define

$$\mathrm{Sw}_{G_{U',x}} := \mathrm{Sw}_{G_{U',x'}} \otimes_{\mathbb{Z}_\ell[G_{U',x'}]} \mathbb{Z}_\ell[G_{U'}].$$

and we denote by

$$\mathrm{sw}_{G_{U',x}} : G_{U'} \rightarrow \mathbb{Z}_\ell$$

its character.

Proposition 9.5. *In the situation above we have the following properties:*

- (a) *The definition of $\mathrm{Sw}_{G_{U',x}}$ is independent of the choice of x'/x .*
- (b)

$$\mathrm{sw}_{G_{U',x}}(\sigma) = \begin{cases} \sum_{\substack{y/x \\ \sigma(y)=y}} (1 - i_{G_{U',y}}(\sigma)), & \text{if } \sigma \neq 1, \\ \left(\sum_{y/x} (1 + v_y(\mathfrak{D}_{C'/C})) \right) - |G_{U'}|, & \text{if } \sigma = 1, \end{cases}$$

where $y \in C'$ maps to x , $i_{G_{U',y}} : G_{U',y} \rightarrow \mathbb{Z}$ is the function defined in (3.5) and $v_y(\mathfrak{D}_{C'/C})$ is the multiplicity at y of the different $\mathfrak{D}_{B/\mathcal{O}_{C,x}}$, where B is the integral closure of $\mathcal{O}_{C,x}$ in K' , see Definition 3.18.

- (c) *Let R be a complete DVR which is finite over \mathbb{Z}_ℓ with maximal ideal \mathfrak{m} and residue field \mathbb{F}_λ . Let \mathcal{F} be a free lisse R -sheaf and assume that $(\mathcal{F} \otimes_R \mathbb{F}_\lambda)|_{U'}$ is trivial, i.e. we can view $\mathcal{F}_{\bar{\eta}} \otimes_R \mathbb{F}_\lambda$ as a $\mathbb{F}_\lambda[G_{U'}]$ -module. Then*

$$\mathrm{Swan}_x(\mathcal{F}) = \mathrm{Swan}_x(\mathcal{F} \otimes_R \mathbb{F}_\lambda) = \dim_{\mathbb{F}_\lambda} \mathrm{Hom}_{\mathbb{F}_\lambda[G_{U'}]}(\mathrm{Sw}_{G_{U',x}} \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\lambda, \mathcal{F}_{\bar{\eta}} \otimes_R \mathbb{F}_\lambda).$$

- (d) *Let \mathcal{F} be a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on U such that $\mathcal{F}|_{U'}$ is trivial, i.e. $\mathcal{F}_{\bar{\eta}}$ is a $\bar{\mathbb{Q}}_\ell[G_{U'}]$ -module. Then for all closed points $x \in C$*

$$\mathrm{Swan}_x(\mathcal{F}) = \frac{1}{|G_{U'}|} \sum_{\sigma \in G_{U'}} \mathrm{sw}_{G_{U',x}}(\sigma) \cdot \mathrm{Tr}(\sigma | \mathcal{F}_{\bar{\eta}}).$$

Proof. Denote by $\iota_{x'} : G_{U',x'} \hookrightarrow G_{U'}$ the inclusion. By Theorem 4.63, (b), to prove (a) it suffices to check that the character of

$$\mathrm{Ind}_{\iota_{x'}}(\mathrm{Sw}_{G_{U',x'}}) = \mathrm{Sw}_{G_{U',x'}} \otimes_{\mathbb{Z}_\ell[G_{U',x'}]} \mathbb{Z}_\ell[G_{U'}]$$

is independent of the choice of x' . Hence it suffices to prove (b). To this end, notice that by Lemma 4.38 the character of $\mathrm{Ind}_{\iota_{x'}}(\mathrm{Sw}_{G_{U',x'}})$ is given by

$$G_{U'} \ni \sigma \mapsto \chi(\sigma) := \frac{1}{|G_{U',x'}|} \sum_{\substack{\tau \in G_{U'} \\ \tau \sigma \tau^{-1} \in G_{U',x'}}} \mathrm{sw}_{G_{U',x'}}(\tau \sigma \tau^{-1}),$$

where $\mathrm{sw}_{G_{U',x'}}$ is the Swan character of $G_{U',x'}$, see Definition 4.45.

First case: $\sigma \neq 1$. We have

$$\tau \sigma \tau^{-1} \in G_{U',x'} \iff \sigma \in G_{U',\tau^{-1}(x')} = \tau^{-1} \circ G_{U',x'} \circ \tau$$

and

$$\mathrm{sw}_{G_{U',x'}}(\tau \sigma \tau^{-1}) = \mathrm{sw}_{G_{U',\tau^{-1}(x')}}(\sigma).$$

Since $G_{U'}$ acts transitively on the points over x , we have a bijection of sets $G_{U'}/G_{U',x'} \xrightarrow{1:1} \{y/x\} \subseteq C'$. Hence

$$\chi(\sigma) = \sum_{\substack{y/x \\ \sigma(y)=y}} \text{sw}_{G_{U',y}}(\sigma).$$

The formula in (b) for $\sigma \neq 1$ thus follows from the definition of $\text{sw}_{G_{U',y}}$, see Definition 4.45. (Notice that since $k = \bar{k}$ we have $G_{U',x'} = G_{U',x'}^0$.)

Second case: $\sigma = 1$. In this case

$$\begin{aligned} \chi(1) &= \frac{|G_{U'}|}{|G_{U',x'}|} \cdot \text{sw}_{G_{U',x'}}(1) = \sum_{y/x} \text{sw}_{G_{U',y}}(1) \\ &= \sum_{y/x} \left(\left(\sum_{\sigma \neq 1} i_{G_{U',y}}(\sigma) \right) - (|G_{U',y}| - 1) \right). \end{aligned}$$

Since $\sum_{\sigma \neq 1} i_{G_{U',y}}(\sigma) = v_y(\mathfrak{D}_{C'/C})$ (see Proposition 3.25) and $\sum_{y/x} |G_{U',y}| = |G_{U'}|$ we obtain the formula for (b).

The formula in (c) follows directly from $\text{Swan}_x(\mathcal{F}) = b(\text{Res}_{\iota_{x'}}(\mathcal{F}_{\bar{\eta}}))$ (see Theorem 4.85), the definition of $b(-)$ in Definition 4.71 and that $\text{Ind}_{\iota_{x'}}$ is left adjoint to $\text{Res}_{\iota_{x'}}$, see (4.1).

Finally, let us prove (d). First notice that if P and V are two finite dimensional $\bar{\mathbb{Q}}_\ell$ -representations of $G_{U'}$, then

$$\text{Hom}_{\bar{\mathbb{Q}}_\ell[G_{U'}]}(P, V) = (P^\vee \otimes_{\bar{\mathbb{Q}}_\ell} V)^{G_{U'}},$$

see Example 4.1. The $\bar{\mathbb{Q}}_\ell$ -dimension of the right-hand side is equal to the number of times the trivial representation appears in $P^\vee \otimes_{\bar{\mathbb{Q}}_\ell} V$. Let χ_P (resp. χ_V) be the character of P (resp. V); the character of P^\vee is given by $\chi_{P^\vee}(\sigma) = \chi_P(\sigma^{-1})$. The character of $P^\vee \otimes_{\bar{\mathbb{Q}}_\ell} V$ is given by $\chi_{P^\vee} \cdot \chi_V$. Hence by Corollary 4.26 we obtain

$$\dim_{\bar{\mathbb{Q}}_\ell} \text{Hom}_{\bar{\mathbb{Q}}_\ell[G_{U'}]}(P, V) = \frac{1}{|G_{U'}|} \sum_{\sigma \in G_{U'}} \chi_P(\sigma^{-1}) \cdot \chi_V(\sigma).$$

Therefore the statement for (d) follows from the formula (cf. (c))

$$\text{Swan}_x(\mathcal{F}) = \dim_{\bar{\mathbb{Q}}_\ell} \text{Hom}_{\bar{\mathbb{Q}}_\ell[G_{U'}]}(\text{Sw}_{G_{U',x}} \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell, \mathcal{F}_{\bar{\eta}}),$$

the discussion above and $\text{sw}_{G_{U'}}(\sigma) = \text{sw}_{G_{U'}}(\sigma^{-1})$, see (b). \square

Lemma 9.6. *In the situation of 9.1.3 denote by $\Delta_{C'} \subseteq C' \times_k C'$ the diagonal and by Γ_σ the graph of σ acting on C' . Then Γ_σ and $\Delta_{C'}$ intersect properly and*

$$(\Gamma_\sigma \cdot \Delta_{C'}) = \sum_{x \in C' \setminus U} \sum_{\substack{x'/x \\ \sigma(x')=x'}} i_{G_{U',x'}}(\sigma), \quad \text{for all } \sigma \neq 1,$$

where the left-hand side is defined as in (8.1).

Proof. Take $\sigma \in G_{U'} \setminus \{1\}$ and $x' \in C'$ with $\sigma(x') = x'$. (Notice that this implies $x' \in C' \setminus U'$ since $G_{U'}$ acts transitively and freely on the fibers over U .) We have to show

$$i_{G_{U',x'}}(\sigma) = \text{length}(\mathcal{O}_{C',x'}/I_\sigma),$$

where $I_\sigma \subseteq \mathcal{O}_{C',x'}$ is the ideal generated by the elements $a - \sigma^*(a)$, $a \in \mathcal{O}_{C',x'}$. Denote by x the image of x' in C . Denote by A the completion of $\mathcal{O}_{C,x}$ and by A' the completion of $\mathcal{O}_{C',x'}$. Then the right-hand side of the equality above is equal to $\text{length}(A'/I_\sigma \cdot A')$. Further by Theorem 3.24 there exists a generator of A'/A , i.e. an element $\alpha \in A'$ such that $A' = A[\alpha]$. It follows that I_σ is the ideal generated by $\alpha - \sigma^*(\alpha)$. Thus by definition: $\text{length}(A'/I_\sigma \cdot A') = v_{x'}(\alpha - \sigma^*(\alpha)) = i_{G_{U',x'}}(\sigma)$. \square

Notation 9.7. Let D be a smooth projective curve over an algebraically closed field and $V \subseteq D$ a strict non-empty open subset. In Corollary 9.4, we defined $\chi_c(V, A)$ and in part (b) of this corollary we saw that it is independent of the choice of A . We set

$$\chi_c(V) := \chi_c(V, A), \quad \chi(D) := \chi_c(V) + \text{card}(D \setminus V)$$

where A is some ℓ -adic coefficient ring. We have an exact sequence $\cdots \rightarrow H^i(D, A) \rightarrow H^i(D \setminus V, A) \rightarrow H_c^{i+1}(V, A) \rightarrow \cdots$. This shows that if A is a field then $\chi(D) = \sum_i (-1)^i \dim_A H^i(D, A)$.

Lemma 9.8. *In the situation of 9.1.3 we have*

$$\chi(C') = 2 - 2g(C') = |G_{U'}| \cdot \chi(C) - \sum_{x' \in C' \setminus U'} v_{x'}(\mathfrak{D}_{C'/C}),$$

where $v_{x'}(\mathfrak{D}_{C'/C})$ is as in Proposition 9.5, (b) and $g(C')$ is the genus of the connected smooth projective curve C' . In particular, $\chi_c(U') = 2 - 2g(C') - \text{card}(C' \setminus U')$.

Proof. We first prove $2 - 2g(C) = \chi(C)$. Let S be a complete local integral ring with residue field k and fraction field E of characteristic 0, which embeds into the complex numbers \mathbb{C} (e.g. $S = W(k)$ the ring of Witt vectors of k). By [20, III, Cor 7.4] there exists a smooth projective curve \mathcal{C} over S such that $\mathcal{C} \otimes_S k = C$. We have a morphism $\text{Spec } \mathbb{C} \rightarrow \text{Spec } S$ and by [12, Arcata V, Th (3.1)] the induced cospecialization map

$$H^i(\mathcal{C}_{\mathbb{C}}, \mathbb{Z}/\ell^n) \xrightarrow{\sim} H^i(\bar{\mathcal{C}}, \mathbb{Z}/\ell^n)$$

is an isomorphism. Using Remark 8.3 it follows that $\chi(C)$ is the Euler characteristic of the complex projective curve $\mathcal{C}(\mathbb{C})$, which is equal to $2 - 2g(\mathcal{C}_{\mathbb{C}})$. Since the genus stays constant on the fibers of flat families of curves we have $g(\mathcal{C}_{\mathbb{C}}) = g(C)$. Now the lemma follows directly from the Hurwitz genus formula (see e.g. [21, IV, Cor 2.4]). \square

Proof of Theorem 9.1. Given a closed point $x \in C$ the number of closed points $\bar{x} \in C \otimes_k \bar{k}$ lying over x is $[k(x) : k]$. Hence by Lemma 9.2 it suffices to consider the case $k = \bar{k}$. Let \mathcal{F} be a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on U . By Theorem 7.13 and Lemma 4.66 we find a finite field extension E/\mathbb{Q}_ℓ with ring of integers R/\mathbb{Z}_ℓ and a free lisse R -sheaf \mathcal{F}' such that $\mathcal{F} = (\mathcal{F}' \otimes_R E) \otimes_E \bar{\mathbb{Q}}_\ell$. Thus by 9.1.1, (a) and Corollary 9.4, (b) it suffices to prove the following: Let \mathbb{F}_λ be a finite field of characteristic ℓ and \mathcal{F} a lisse \mathbb{F}_λ -sheaf of rank r , then

$$\chi_c(U, \mathcal{F}) = r \cdot \chi_c(U) - \sum_{x \in C \setminus U} \text{Swan}_x(\mathcal{F}). \quad (9.3)$$

The sheaf \mathcal{F} corresponds to a homomorphism $\pi_1(U, \bar{\eta}) \rightarrow \mathrm{GL}_r(\mathbb{F}_\lambda)$; since its target is a finite group we find a connected finite étale Galois cover $\pi : U' \rightarrow U$ with Galois group $G_{U'}$ which trivializes \mathcal{F} .

Let A be an ℓ -adic coefficient ring. By Theorem 7.13 we can identify the category of free lisse A -sheaves on U which are trivial on U' with the category of finitely generated $A[G_{U'}]$ -modules which are free as A -modules. We can restrict the functors C^i from Lemma (b) to this category and obtain functors

$$T^i : (\text{fin. generated } A\text{-free } A[G_{U'}]\text{-modules}) \rightarrow (\text{free } A\text{-modules})$$

with a natural transformation $T^1 \rightarrow T^2$, such that for lisse A -sheaves \mathcal{G} which are trivialized by $\pi : U' \rightarrow U$ we get

$$H_c^i(U, \mathcal{G}) = H^i(T^1(\mathcal{G}_{\bar{\eta}}) \rightarrow T^2(\mathcal{G}_{\bar{\eta}})), \quad i \geq 0.$$

We define a function

$$\nu : (\text{fin. generated } A\text{-free } A[G_{U'}]\text{-modules}) \rightarrow \mathbb{Z} \quad (9.4)$$

by

$$\nu_A(M) := \mathrm{rk}_A(T^2(M)) - \mathrm{rk}_A(T^1(M)) - (\mathrm{rk}_A(M) \chi_c(U, \bar{\mathbb{Q}}_\ell) - \sum_{x \in C \setminus U} \mathrm{Swan}_x(M)).$$

Denote by $R_A(G_{U'})$ the Grothendieck group of the category on the left-hand side of (9.4) (see Definition 4.6). Then it follows from 9.1.1, (d) and Corollary 9.4, (c), that ν induces a well-defined group homomorphism

$$\nu_A : R_A(G_{U'}) \rightarrow \mathbb{Z}.$$

Since we reduced to showing (9.3), we have to prove that $\nu_{\mathbb{F}_\lambda} = 0$. Let E be a finite extension of \mathbb{Q}_ℓ with ring of integers R and residue field \mathbb{F}_λ . By Proposition 4.61 there is a homomorphism $d : R_E(G_{U'}) \rightarrow R_{\mathbb{F}_\lambda}(G_{U'})$. Recall that d is constructed as follows: Let V be an $E[G_{U'}]$ -module, which is finite dimensional as an E -vector space. Take $M \subseteq V$ an $R[G_{U'}]$ -submodule which is an R -lattice in V . Then $d([V]) = [M \otimes_R \mathbb{F}_\lambda]$. It follows from 9.1.1, (a) and Corollary 9.4, (b) that we obtain a commutative diagram

$$\begin{array}{ccc} R_E(G_{U'}) & & \\ \downarrow d & \searrow \nu_E & \\ & \mathbb{Z} & \\ & \nearrow \nu_{\mathbb{F}_\lambda} & \\ R_{\mathbb{F}_\lambda}(G_{U'}) & & \end{array}$$

By Theorem 4.63, (a) the map d is surjective. Hence it suffices to show that $\nu_E = 0$. Retranslating this into lisse E -sheaves, we see that it suffices to prove equality (9.3) for \mathcal{F} a lisse E -sheaf of rank r which is trivialized by $\pi : U' \rightarrow U$. Set $V = \mathcal{F}_{\bar{\eta}}$. It is an E -vector space of dimension $\dim_E V = r$ together with a $G_{U'}$ -action. We have (see Theorem 7.13)

$$\pi^* \mathcal{F} \cong V_{U'}, \quad \mathcal{F} \cong (\pi_* V_{U'})^{G_{U'}},$$

where $V_{U'}$ is the constant lisse E -sheaf on U' defined by V . Denote by $\bar{\pi} : C' \rightarrow C$ the unique finite morphism between smooth proper curves over

k , which restricts to $\pi : U' \rightarrow U$, and by $j : U \hookrightarrow C$, $j' : U' \hookrightarrow C'$ the open immersions. We obtain (via a direct computation)

$$j_! \mathcal{F} \cong j_!(\pi_* V_{U'})^{G_{U'}} \cong (j_! \pi_* V_{U'})^{G_{U'}} \cong (\bar{\pi}_* j'_! V_{U'})^{G_{U'}}.$$

By Lemma 8.8 and 8.1.2, (c) we get

$$H_c^i(U, \mathcal{F}) = H^i(C, (\bar{\pi}_* j'_! V_{U'})^{G_{U'}}) = H^i(C, \bar{\pi}_* j'_! V_{U'})^{G_{U'}} = H_c^i(U', V_{U'})^{G_{U'}}.$$

Next, if R is the ring of integers of E , then we find an R -lattice $M \subseteq V$ with $G_{U'}$ -action such that $V \cong M \otimes_R E$ as $E[G_{U'}]$ -modules. If $(R/\mathfrak{m}^n)_{U'} \rightarrow \mathcal{I}^\bullet$ is an injective resolution of the constant sheaf R/\mathfrak{m}^n on U' , then $\mathcal{I}^\bullet \otimes_{R/\mathfrak{m}^n} (M \otimes_R R/\mathfrak{m}^n)_{U'}$ is a resolution by injectives of the constant sheaf $(M \otimes_R R/\mathfrak{m}^n)$ which is compatible with the $G_{U'}$ -action. This gives a $G_{U'}$ -equivariant isomorphism $H_c^i(U', V_{U'}) \cong H_c^i(U', E_{U'}) \otimes_E V$. All together we obtain

$$H_c^i(U, \mathcal{F}) = (H_c^i(U', E_{U'}) \otimes_E V)^{G_{U'}}.$$

The dimension of the E -vector space $(H_c^i(U', E_{U'}) \otimes_E V)^{G_{U'}}$ is equal to the number of times the trivial rank one representation is contained in the $E[G_{U'}]$ -module $H_c^i(U', E_{U'}) \otimes_E V$. Hence Corollary 4.26 yields

$$\begin{aligned} \chi_c(U, \mathcal{F}) &= -\dim_E(H_c^1(U', E) \otimes_E V)^{G_{U'}} + \dim_E(H_c^2(U', E) \otimes_E V)^{G_{U'}} \\ &= \frac{1}{|G_{U'}|} \sum_{\sigma \in G_{U'}} \text{Tr}(\sigma|V) (\text{Tr}(\sigma^*|H_c^2(U', E)) - \text{Tr}(\sigma^*|H_c^1(U', E))). \end{aligned} \quad (9.5)$$

Set $Y' := C' \setminus U'$. The exact sequence $\cdots \rightarrow H^i(C', E) \rightarrow H^i(Y', E) \rightarrow H_c^{i+1}(U', E) \rightarrow \cdots$ from 8.1.2, (d) yields a short exact sequence

$$0 \rightarrow H^0(C', E) \rightarrow H^0(Y', E) \rightarrow H_c^1(U', E) \rightarrow H^1(C', E) \rightarrow 0$$

and an isomorphism

$$H_c^2(U', E) \cong H^2(C', E).$$

For $\sigma \in G_{U'}$ we obtain

$$\begin{aligned} &\text{Tr}(\sigma^*|H_c^2(U', E)) - \text{Tr}(\sigma^*|H_c^1(U', E)) \\ &= -\text{Tr}(\sigma^*|H^0(Y', E)) + \sum_{i=0}^2 (-1)^i \text{Tr}(\sigma^*|H^i(C', E)). \end{aligned} \quad (9.6)$$

We can write

$$H^0(Y', E) = \bigoplus_{x' \in Y'} E_{x'},$$

where $E_{x'} = E$ and $\sigma \in G_{U'}$ acts via:

$$\sigma^* : E_{x'} = E \xrightarrow{\text{id}} E = E_{\sigma(x')}.$$

Thus

$$\text{Tr}(\sigma^*|H^0(Y', E)) = \sum_{x \in Y} \sum_{\substack{x'/x \\ \sigma(x')=x'}} 1, \quad \sigma \in G_{U'}. \quad (9.7)$$

Further, for $\sigma \neq 1$ the graph Γ_σ and the diagonal $\Delta_{C'}$ intersect properly, see Lemma 9.6. Hence the Lefschetz trace formula (Theorem 8.7) yields

$$\sum_{i=0}^2 (-1)^i \operatorname{Tr}(\sigma^* | H^i(C', E)) = \begin{cases} (\Gamma_\sigma \cdot \Delta_{C'}), & \text{if } \sigma \neq 1, \\ \chi(C'), & \text{if } \sigma = 1. \end{cases} \quad (9.8)$$

By Lemma 9.8, equation (9.7) and Proposition 9.5, (b) we get

$$\begin{aligned} & -\operatorname{Tr}(1^* | H^0(Y', E)) + \sum_{i=0}^2 (-1)^i \operatorname{Tr}(1^* | H^i(C', E)) \\ &= -\operatorname{card}(Y') + |G_{U'}| \cdot (\chi_c(U) + \operatorname{card}(Y)) - \sum_{x' \in Y'} v_{x'}(\mathfrak{D}_{C'/C}) \\ &= |G_{U'}| \cdot \chi_c(U) - \sum_{x \in Y} \left(\sum_{x'/x} 1 + v_{x'}(\mathfrak{D}_{C'/C}) \right) - |G_{U'}| \\ &= |G_{U'}| \cdot \chi_c(U) - \sum_{x \in Y} \operatorname{sw}_{G_{U',x}}(1). \end{aligned}$$

By (9.7), (9.8), Lemma 9.6 and Proposition 9.5, (b), we have for $\sigma \in G_{U'} \setminus \{1\}$

$$\begin{aligned} & -\operatorname{Tr}(\sigma^* | H^0(Y', E)) + \sum_{i=0}^2 (-1)^i \operatorname{Tr}(\sigma^* | H^i(C', E)) \\ &= -\sum_{x \in Y} \sum_{\substack{x'/x \\ \sigma(x')=x'}} 1 + \sum_{x \in Y} \sum_{\substack{x'/x \\ \sigma(x')=x'}} i_{G_{U',x'}}(\sigma) \\ &= \sum_{x \in Y} \sum_{\substack{x'/x \\ \sigma(x')=x'}} (i_{G_{U',x'}}(\sigma) - 1) \\ &= -\sum_{x \in Y} \operatorname{sw}_{G_{U',x}}(\sigma). \end{aligned}$$

Hence by (9.5), (9.6) and Proposition 9.5, (d)

$$\begin{aligned} \chi_c(U, \mathcal{F}) &= \frac{1}{|G_{U'}|} \cdot \operatorname{Tr}(1^* | V) \cdot \left(|G_{U'}| \cdot \chi_c(U) - \sum_{x \in Y} \operatorname{sw}_{G_{U',x}}(1) \right) \\ &\quad - \frac{1}{|G_{U'}|} \cdot \sum_{\sigma \in G_{U'} \setminus \{1\}} \left(\operatorname{Tr}(\sigma | V) \cdot \sum_{x \in Y} \operatorname{sw}_{G_{U',x}}(\sigma) \right) \\ &= r \cdot \chi_c(U) - \sum_{x \in Y} \operatorname{Swan}_x(\mathcal{F}). \end{aligned}$$

This finishes the proof. \square

Example 9.9. Let k be an algebraically closed field of characteristic $p > 0$, $q = p^n$, m a natural number with $(m, p) = 1$ and $\psi : \mathbb{F}_q \rightarrow \mathbb{Q}_\ell^\times$ a group homomorphism. Let $\mathcal{L}_{m,\psi}$ be the lisse rank 1 sheaf on \mathbb{A}_k^1 from Example 7.16. Denote by ∞ the point in the complement of $\mathbb{A}_k^1 \subseteq \mathbb{P}_k^1$. Then $\operatorname{Swan}_\infty(\mathcal{L}_{m,\psi}) = m$, by Example 4.87, and $\chi_c(\mathbb{A}_k^1) = 1$ by Lemma 9.8. Thus the Grothendieck-Ogg-Shafarevich formula gives

$$\chi_c(\mathbb{A}_k^1, \mathcal{L}_\psi) = 1 - m.$$

9.2. Cohomological description of the Swan representation in equal characteristic. We fix the following situation:

- $k = \bar{k}$ is an algebraically closed field of positive characteristic p .
- We write $\mathbb{G}_m = \mathbb{G}_{m,k} = \operatorname{Spec} k[t, \frac{1}{t}]$ and denote by $\bar{\eta} \rightarrow \mathbb{G}_m$ a geometric point over the generic point. We embed $\mathbb{G}_m \subseteq \mathbb{P}^1$ and write $\mathbb{P}^1 \setminus \mathbb{G}_m = \{0, \infty\}$.
- K is a complete discrete valuation field containing k with the property that k maps isomorphically onto the residue field of K . The choice of a local parameter of K induces an isomorphism of K with the completion of $k(\mathbb{P}^1)$ at ∞ , i.e. $K \cong k((t^{-1}))$. We fix such an isomorphism.
- Let L/K be a finite Galois extension with Galois group G .

In the following we explain Katz' cohomological description of Sw_G (see Theorem 4.53).

9.2.1. Let A be an ℓ -adic coefficient ring (see Convention 7.12). Let \mathcal{F} be a lisse A -sheaf on \mathbb{G}_m with finite monodromy, i.e., the image of the corresponding representation $\pi_1(\mathbb{G}_m, \bar{\eta}) \rightarrow \operatorname{Aut}(\mathcal{F}_{\bar{\eta}})$ is finite. Following Katz, we say that \mathcal{F} is *special* if it is tame at 0 and if its monodromy group has a unique p -Sylow group. Then Katz constructs in [24, Cor. 1.5.7] an equivalence of categories

$$(-)^{\operatorname{can}} : (\text{fin. gen. } A[G] \text{ -- modules}) \xrightarrow{\sim} (\text{special lisse } A \text{ -- sheaves on } \mathbb{G}_m). \quad (9.9)$$

If M is a finitely generated $A[G]$ -module, then the sheaf M^{can} is called the *canonical extension* of M . The name is justified by the fact that restricting the representation corresponding to M^{can} to the decomposition group at ∞ gives back the original $A[G]$ -module M . (Here we view G as a quotient of the absolute Galois group of $k((t^{-1}))$ via the isomorphism $K \cong k((t^{-1}))$ fixed above.) The functor $(-)^{\operatorname{can}}$ commutes with direct sums, tensor products, duals, and with changes of coefficients $A \rightarrow A'$.

Theorem 9.10 ([24, Thm. 1.6.8]). *Denote by $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ the canonical open immersion. Then with the notation from Theorem 4.53 there is an isomorphism of $\mathbb{Z}_\ell[G]$ -modules*

$$H^1(\mathbb{A}^1, j_!(\mathbb{Z}_\ell[G]^{\operatorname{can}})) \cong \operatorname{Sw}_G. \quad (9.10)$$

Here the G -module structure on the left-hand side is defined as follows: The left G -module $\mathbb{Z}_\ell[G]$ defines the constructible sheaf $j_!(\mathbb{Z}_\ell[G]^{\operatorname{can}})$ and the right G -module structure of $\mathbb{Z}_\ell[G]$ defines via functoriality a G -action on the sheaf $j_!(\mathbb{Z}_\ell[G]^{\operatorname{can}})$, which via functoriality induces a G -action on the left-hand side of (9.10).

Proof. We sketch the proof following Katz. For an arbitrary ℓ -adic coefficient ring A we define the functor

$$T : (\text{fin. gen. } A[G] \text{ -- modules}) \rightarrow (\text{fin. gen. } A \text{ -- modules}) \quad (9.11)$$

by

$$T(M) := H^1(\mathbb{A}^1, j_!(M^{\operatorname{can}})).$$

We have $H^i(\mathbb{A}^1, j_!(M^{\operatorname{can}})) = 0$, for $i \neq 1$. (This is clear for $i = 0$ and follows for $i \geq 2$ from 8.1.2, (e).) Hence T is an exact functor. Furthermore, $A[G]$ acts

left- $A[G]$ -linearly on itself by right multiplication. Thus $T(A[G])$ becomes an $A[G]$ -module by functoriality. Using an argument similar to the proof of Lemma 9.3 one can show:

- (a) $T(M) = T(A[G]) \otimes_{A[G]} M$, for all finitely generated $A[G]$ -modules M .
- (b) $T(A[G])$ is a finitely generated projective $A[G]$ -module.
- (c) $T(A[G]) \otimes_A A' = T(A'[G])$, where $A \rightarrow A'$ is any map of ℓ -adic coefficient rings.

By (b), (c) and the uniqueness part of Theorem 4.53, (b), it suffices to show that $T(\bar{\mathbb{Q}}_\ell[G])$ is the Swan representation. By Proposition 4.25, (a) and Corollary 4.26, it suffices to show that for any irreducible $\bar{\mathbb{Q}}_\ell[G]$ -module W we have

$$\dim_{\bar{\mathbb{Q}}_\ell}(T(\bar{\mathbb{Q}}_\ell[G]) \otimes_{\bar{\mathbb{Q}}_\ell} W^\vee)^G = \dim_{\bar{\mathbb{Q}}_\ell}(\text{Sw}_G \otimes_{\mathbb{Z}_\ell} W^\vee)^G. \quad (9.12)$$

Since dualizing induces a bijection on the set of irreducible representations we may replace W^\vee by W . The Swan representation is self dual (as one easily checks that $\text{sw}_G(g) = \text{sw}_G(g^{-1})$). Hence

$$\dim_{\bar{\mathbb{Q}}_\ell}(\text{Sw}_G \otimes_{\mathbb{Z}_\ell} W)^G = \dim_{\bar{\mathbb{Q}}_\ell} \text{Hom}_{\bar{\mathbb{Q}}_\ell[G]}(\text{Sw}_G \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell, W) = \text{Swan}(W),$$

where the second equality follows from Theorem 4.85 and Remark 4.72. On the other hand by (a) above

$$\begin{aligned} \dim_{\bar{\mathbb{Q}}_\ell}(T(\bar{\mathbb{Q}}_\ell[G]) \otimes_{\bar{\mathbb{Q}}_\ell} W)^G &= \dim_{\bar{\mathbb{Q}}_\ell}(T(\bar{\mathbb{Q}}_\ell[G]) \otimes_{\bar{\mathbb{Q}}_\ell[G]} W) \\ &= \dim_{\bar{\mathbb{Q}}_\ell} H^1(\mathbb{A}^1, j_! W^{\text{can}}) \\ &= -\chi(\mathbb{A}^1, j_! W^{\text{can}}). \end{aligned}$$

Furthermore, $\chi(\mathbb{A}^1, j_! W^{\text{can}}) = \chi_c(\mathbb{A}^1, j_! W^{\text{can}}) = \chi_c(\mathbb{G}_m, W^{\text{can}})$, where the first equality follows e.g. from [25, 2.0.7] and the second equality holds by definition. We have $\chi_c(\mathbb{G}_m, \bar{\mathbb{Q}}_\ell) = 0$. Since W^{can} is lisse on \mathbb{G}_m and tame in 0, the formula of Grothendieck-Ogg-Shafarevich yields

$$\chi_c(\mathbb{G}_m, W^{\text{can}}) = -\text{Swan}_\infty(W^{\text{can}}) = -\text{Swan}(W).$$

Putting everything together yields the theorem. \square

10. HIGHER DIMENSIONAL RAMIFICATION THEORY VIA CURVES

In this section we present the ramification theory of lisse sheaves on a higher dimensional smooth variety over a perfect field of positive characteristic, following [16, 3].

10.0.1. Throughout this section we fix the following notation:

- k is a perfect field of characteristic $p > 0$ and \bar{k} an algebraic closure.
- ℓ is a prime number different from p .
- A k -scheme is a scheme which is separated and of finite type over k .
- For a k -scheme X we denote by $\text{Cu}(X)$ the set of normalizations of closed integral 1-dimensional subschemes of X . In other words, $C \in \text{Cu}(X)$ is a smooth connected curve over k with a morphism $\nu : C \rightarrow X$, which is birational onto its image.

- If C is a smooth connected k -scheme of dimension 1, we denote by \widehat{C} the (up to isomorphism) unique smooth projective curve over k , which admits an open dense embedding $C \hookrightarrow \widehat{C}$.

10.1. Lisse sheaves with bounded ramification.

Definition 10.1. Let U be a normal k -scheme. A *divisorial compactification*³ of U is a normal proper k -scheme X together with an open dense immersion $j : U \hookrightarrow X$ such that the complement $X \setminus U$ is the support of an effective Cartier divisor.

Remark 10.2. Let U be a normal k -scheme.

- (a) A divisorial compactification of U exists. Indeed, by Nagata's compactification theorem (see e.g. [11, Thm. 4.1]), we find a dense open immersion of U into a proper k -scheme Y . Let \tilde{Y} be the blow-up of Y in the closed subscheme $Y \setminus U$ (with its reduced scheme structure). We obtain a dominant open immersion $U \hookrightarrow \tilde{Y}$ whose complement is the support of the exceptional divisor E of the blow-up, which is an effective Cartier divisor. Now let $\nu : X \rightarrow \tilde{Y}$ be the normalization of \tilde{Y} . Then ν is a finite morphism which is an isomorphism over U . We obtain an open dense embedding $U \hookrightarrow X$ whose complement is the support of the effective Cartier divisor ν^*E . Then $U \hookrightarrow X$ is a divisorial compactification of U .
- (b) If C is a smooth connected curve over k , then with the notation from 10.0.1 the inclusion $C \hookrightarrow \widehat{C}$ is the (up to isomorphism) unique divisorial compactification of C .
- (c) Let X be a divisorial compactification of U and $C \in \text{Cu}(U)$. Then by the valuative criterion for properness, the morphism $\nu : C \rightarrow U$ extends uniquely to a morphism $\nu_X : \widehat{C} \rightarrow X$. Furthermore \widehat{C} is the normalization of its image in X , hence $\widehat{C} \in \text{Cu}(X)$.

Definition 10.3. Let C be a smooth connected curve over k , A an ℓ -adic coefficient ring in the sense of Convention 7.12 and \mathcal{F} a lisse A -sheaf on C . We define the *Swan conductor* of \mathcal{F} to be the following effective Cartier divisor on \widehat{C} :

$$\text{Swan}(\mathcal{F}) := \sum_{x \in \widehat{C}} \text{Swan}_x(\mathcal{F}) \cdot [x].$$

Here $\text{Swan}_x(\mathcal{F})$ is the Swan conductor of \mathcal{F} on C at x , see 9.1.1. Notice that $\text{Swan}_x(\mathcal{F}) \in \mathbb{Z}_{\geq 0}$ by Theorem 4.85.

Definition 10.4. Let U be a normal k -scheme, $U \hookrightarrow X$ a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. Let A be an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on U . Then we say that the *ramification of \mathcal{F} is bounded by D* if the following condition is satisfied:

$$\text{Swan}(\nu^* \mathcal{F}) \leq \nu_X^* D, \quad \text{for all } C \in \text{Cu}(U). \quad (10.1)$$

Here $\nu : C \rightarrow U$ is the natural map and $\nu_X : \widehat{C} \rightarrow X$ its unique extension; the inequality takes place in the monoid of effective divisors on \widehat{C} .

³This is non-standard terminology.

Example 10.5. Let $\mathcal{L}_{m,\psi}$ be the lisse sheaf on $\mathbb{A}^1 \subseteq \mathbb{P}^1$ from Example 7.16. Then the ramification of $\mathcal{L}_{m,\psi}$ is bounded by $m \cdot \{\infty\}$, by Example 4.87.

Remark 10.6. In the situation above we find a DVR R which is finite over \mathbb{Z}_ℓ and a lisse R -sheaf $(\mathcal{F}_n)_n$ on U such that A is an R -algebra and $\mathcal{F} \cong (\mathcal{F}_n) \otimes_R A$. Then for any $\nu : C \rightarrow U$ in $\text{Cu}(U)$ we have $\nu^* \mathcal{F} \cong (\nu^* \mathcal{F}_n) \otimes_R A$ and by 9.1.1, (a)

$$\text{Swan}(\nu^* \mathcal{F}) = \text{Swan}(\nu^* \mathcal{F}_1).$$

Thus whether the ramification of \mathcal{F} is bounded or not by a given Cartier divisor D depends only on \mathcal{F}_1 .

Remark 10.7. In the situation of Definition 10.4 if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of lisse A -sheaves on U and the ramification of \mathcal{F}' and \mathcal{F}'' is bounded by D' and D'' , respectively, the ramification of \mathcal{F} is bounded by $D' + D''$. This follows immediately from 9.1.1, (d).

Proposition 10.8. *Let U be a normal k -scheme and $U \hookrightarrow X$ a divisorial compactification. Let A be an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on U . Then there exists an effective Cartier divisor D on X supported in $X \setminus U$ such that the ramification of \mathcal{F} is bounded by D .*

Before we can prove the proposition we need a notion of discriminant in higher dimensions. The following definition is due to Alexander Schmidt.

Definition 10.9. Let $\pi : Y' \rightarrow Y$ be a finite and generically étale morphism of degree n between connected normal k -schemes. Denote by $\text{Tr}_{Y'/Y} : \pi_* \mathcal{O}_{Y'} \rightarrow \mathcal{O}_Y$ the trace map which is induced by $\text{Tr}_{k(Y')/k(Y)}$. We define the discriminant $\mathcal{I}(D_{Y'/Y})$ of Y'/Y to be the sheaf associated to the following presheaf of ideals in \mathcal{O}_Y

$$Y \supseteq U \mapsto \sum_{\{x_1, \dots, x_n\}} \det((\text{Tr}_{Y'/Y}(x_i x_j))_{i,j}) \cdot \mathcal{O}_Y(U) \subseteq \mathcal{O}_Y(U),$$

where the sum is over all subset $\{x_1, \dots, x_n\} \subseteq \mathcal{O}_{Y'}(\pi^{-1}(U))$, which consist of $\mathcal{O}_Y(U)$ -linearly independent elements.

Lemma 10.10. *In the situation of Definition 10.9 the discriminant $\mathcal{I}(D_{Y'/Y})$ has the following properties:*

- (a) $\mathcal{I}(D_{Y'/Y}) \subseteq \mathcal{O}_Y$ is a coherent ideal sheaf.
- (b) If $\pi : Y' \rightarrow Y$ is free and e_1, \dots, e_n is a basis of the \mathcal{O}_Y -module $\pi_* \mathcal{O}_{Y'}$, then $\mathcal{I}(D_{Y'/Y}) = \det((\text{Tr}_{Y'/Y}(e_i e_j))_{i,j}) \cdot \mathcal{O}_Y$.
- (c) Assume $Y = \text{Spec } A$ and $Y' = \text{Spec } B$ are smooth affine curves. Then $\mathcal{I}(D_{Y'/Y})$ is the sheaf induced by the discriminant ideal $\mathfrak{d}_{B/A} \subseteq A$ from Definition 3.18.
- (d) If $U \subseteq Y$ is an open subset such that $\pi^{-1}(U) \rightarrow U$ is étale, then $\mathcal{I}(D_{Y'/Y})|_U = \mathcal{O}_U$.
- (e) Let $\nu : C \rightarrow Y$ be a morphism from a smooth connected curve C to Y and assume that $\nu(C)$ meets the locus over which π is étale. Let C' be the normalization of an irreducible component of $C \times_Y Y'$, which

is generically étale over C . We obtain a commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\nu'} & Y' \\ \pi_{C'} \downarrow & & \downarrow \pi \\ C & \xrightarrow{\nu} & Y. \end{array}$$

Then the image of $\mathcal{I}(D_{Y'/Y})$ under $\nu^* : \nu^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_C$ is contained in $\mathcal{I}(D_{C'/C})$, i.e.

$$\nu^*(\mathcal{I}(D_{Y'/Y})) \subseteq \mathcal{I}(D_{C'/C}).$$

Proof. By definition $\mathcal{I}(D_{Y'/Y})$ is an \mathcal{O}_Y -submodule of \mathcal{O}_Y hence (a) is automatic. Now let $U = \text{Spec } A$ be an affine open subset of Y and write $\pi^{-1}(U) = \text{Spec } B$. Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be two sets of A -linearly independent elements of B and denote $B_x = \sum_i A x_i$, $B_y = \sum_i A y_i$. Assume $B_x \subseteq B_y$ then

$$\det((\text{Tr}_{B/A}(x_i x_j))_{i,j}) \cdot A \subseteq \det((\text{Tr}_{B/A}(y_i y_j))_{i,j}) \cdot A. \quad (10.2)$$

Indeed by assumption we find an $n \times n$ -matrix M with coefficients in A such that (with the obvious vector notation) $x = My$ and we obtain

$$\det((\text{Tr}_{B/A}(x_i x_j))_{i,j}) = \det(M)^2 \cdot \det((\text{Tr}_{B/A}(y_i y_j))_{i,j}),$$

in particular (10.2) holds. This immediately implies (b) and using Lemma 3.21 also (c).

To prove (d) we can assume that B/A is étale and free with basis e_1, \dots, e_n and we have to show $\delta := \det((\text{Tr}_{B/A}(e_i e_j))_{i,j}) \in A^\times$. For this it suffices to show that for any prime $\mathfrak{p} \subseteq A$ the element δ maps to a unit in the residue field $k(\mathfrak{p})$. But since B/A is étale, $B \otimes_A k(\mathfrak{p})$ is étale over $k(\mathfrak{p})$ with basis $\bar{e}_i = e_i \otimes 1$, $i = 1, \dots, n$, and the image of δ in $k(\mathfrak{p})$ equals $\det((\text{Tr}_{B \otimes_A k(\mathfrak{p})/k(\mathfrak{p})}(\bar{e}_i \bar{e}_j))_{i,j})$. It follows from Proposition 3.14, that this element is not zero in $k(\mathfrak{p})$. Hence (d) follows.

Finally, in the situation of (e), we can assume that $Y' \rightarrow Y$ is given by a finite ring extension B/A . If the image of C is a point, then by assumption it lies in the locus over which π is étale and hence $\mathcal{I}(D_{Y'/Y})$ is zero around this point. Thus we can assume that $\nu(C)$ is a curve and therefore $C' \rightarrow C$ corresponds to a finite and generically étale extension of Dedekind domains S/R . Let $x_1, \dots, x_n \in B$ be a sequence of A -linearly independent elements. We have to show that the image of $\det((\text{Tr}_{B/A}(x_i x_j))_{i,j})$ under $A \rightarrow R$ is contained in $\mathfrak{d}_{S/R}$, which amounts to show that

$$\det((\text{Tr}_{B \otimes_A R/R}(\bar{x}_i \bar{x}_j))_{i,j}) \in \mathfrak{d}_{S/R}, \quad (10.3)$$

where \bar{x}_i denotes the image of x_i in $B \otimes_A R$. Observe that all irreducible components of $Y' \times_Y C$ map surjectively to C . Indeed, since $\pi : Y' \rightarrow Y$ is finite and surjective and Y is normal (hence universally unibranch) π is universally open, by [18, Cor. (14.4.4), (i)]. Let $S_1 = S, S_2, \dots, S_r$ be the affine coordinate rings of the normalizations of the irreducible components of $Y' \times_Y C$. We get a natural map $B \otimes_A R \rightarrow \prod_i S_i$ which becomes an isomorphism when tensored with $\otimes_A k(Y)$, since $Y' \times_Y C \rightarrow C$ is generically

étale. It follows that we obtain a commutative diagram

$$\begin{array}{ccc} B \otimes_A R & \longrightarrow & \prod_i S_i \\ & \searrow \text{Tr} & \downarrow \Sigma_i \text{Tr}_{S_i/R} \\ & & R. \end{array}$$

The images of $\bar{x}_1, \dots, \bar{x}_n$ in $\prod_i S_i$ span an R -submodule of rank n . Further taking a basis $\{e_{ij}\}_{j=1, \dots, n_i}$ of S_i/R we get the following basis of $\prod_i S_i$

$$(e_{11}, 0, \dots, 0), \dots, (e_{1, n_1}, 0, \dots, 0), \dots, (0, \dots, 0, e_{r, 1}), \dots, (0, \dots, 0, e_{r, n_r}).$$

Therefore the formula (10.2) (applied with $\{y_i\}$ the above basis) yields

$$\det((\text{Tr}_{B \otimes_A R/R}(\bar{x}_i \bar{x}_j))_{i,j}) \cdot A \subseteq \mathfrak{d}_{S_1/R} \cdots \mathfrak{d}_{S_r/R} \subseteq \mathfrak{d}_{S_1/R}.$$

This proves (10.3) and finishes the proof of the lemma. \square

Proof of Proposition 10.8. We can assume that U is connected; let $\bar{\eta} \rightarrow U$ be a geometric point over the generic point of U . By Remark 10.6 it suffices to consider the case in which A is a finite field of characteristic ℓ . In particular $\mathcal{F}_{\bar{\eta}}$ is finite (as a set). Hence the representation $\pi_1(U, \bar{\eta}) \rightarrow \text{Aut}(\mathcal{F}_{\bar{\eta}})$ defined by \mathcal{F} factors over a finite quotient, i.e. there exists a finite connected étale Galois cover $\pi : U' \rightarrow U$ which trivializes \mathcal{F} . Let $\pi_X : X' \rightarrow X$ be the normalization of X inside the function field of U' . Then π_X is finite, $\pi_X^{-1}(U) = U'$ and X' is a divisorial compactification of U' . Let $\mathcal{I}(D_{X'/X})$ be the discriminant of X'/X (Definition 10.9). Since $\mathcal{I}(D_{X'/X})|_U = \mathcal{O}_U$ we find an effective Cartier divisor D on X with $\mathcal{O}_X(-D) \subseteq \mathcal{I}(D_{X'/X})$. We claim:

$$\text{The ramification of } \mathcal{F} \text{ is bounded by } \text{rk}(\mathcal{F}) \cdot D. \quad (10.4)$$

Take $\nu : C \rightarrow U$ in $\text{Cu}(U)$. Let C' be a connected component of $C \times_U U'$. The projection maps induce maps $C' \rightarrow C$ and $C' \rightarrow U'$, which after compactification yield a commutative diagram

$$\begin{array}{ccc} \widehat{C}' & \xrightarrow{\nu_{X'}} & X' \\ \pi_{\widehat{C}} \downarrow & & \downarrow \pi_X \\ \widehat{C} & \xrightarrow{\nu_X} & X. \end{array}$$

By Lemma 10.10, (e), we have

$$\mathcal{O}_{\widehat{C}}(-D|_{\widehat{C}}) \subseteq \nu_X^* \mathcal{I}(D_{X'/X}) \subseteq \mathcal{I}(D_{\widehat{C}'/\widehat{C}}).$$

Denote by $D_{\widehat{C}'/\widehat{C}}$ the effective divisor on C given by $\mathcal{I}(D_{\widehat{C}'/\widehat{C}})$; it is the discriminant divisor. We obtain

$$\sum_{x'/x} [x' : x] \cdot v_{x'}(\mathfrak{d}_{\widehat{C}'/\widehat{C}}) = \text{mult}_x(D_{\widehat{C}'/\widehat{C}}) \leq \text{mult}_x(D|_{\widehat{C}}),$$

where we use the notation from Proposition 9.5.

On the other hand $C' \rightarrow C$ is a connected finite étale Galois covering, say with Galois group $G_{C'}$, trivializing $\mathcal{E} := \nu^* \mathcal{F}$. Let $\bar{\xi} \rightarrow C$ be a geometric

point over the generic point of C and $x \in \widehat{C} \setminus C$ a closed point. Then by Proposition 9.5, (c) and (b) (see also Remark 4.72)

$$\begin{aligned}
\text{Swan}_x(\mathcal{E}) &= \dim_A \text{Hom}_A[G_{C'}](\text{Sw}_{G_{C'},x} \otimes_{\mathbb{Z}_\ell} A, \mathcal{E}_{\bar{\xi}}) \\
&= \dim_A (\underline{\text{Hom}}_A(\text{Sw}_{G_{C'},x} \otimes_{\mathbb{Z}_\ell} A, \mathcal{E}_{\bar{\xi}})^{G_{C'}}) \\
&\leq \dim_A \underline{\text{Hom}}_A(\text{Sw}_{G_{C'},x} \otimes_{\mathbb{Z}_\ell} A, \mathcal{E}_{\bar{\xi}}) \\
&= \text{rk}(\mathcal{E}) \cdot \text{sw}_{G_{C'},x}(1) \\
&= \text{rk}(\mathcal{F}) \cdot \left(\left(\sum_{x'/x} [x' : x] \left(1 + v_{x'}(\mathfrak{D}_{\widehat{C}'/\widehat{C}}) \right) \right) - |G_{C'}| \right) \\
&\leq \text{rk}(\mathcal{F}) \cdot \sum_{x'/x} [x' : x] \cdot v_{x'}(\mathfrak{D}_{\widehat{C}'/\widehat{C}}) \\
&\leq \text{rk}(\mathcal{F}) \cdot \text{mult}_x(D|_{\widehat{C}})
\end{aligned}$$

Hence $\text{Swan}(\nu^* \mathcal{F}) \leq \text{rk}(\mathcal{F}) \cdot D|_{\widehat{C}}$ which proves the claim (10.4) and the proposition. \square

As a consequence of the Grothendieck-Ogg-Shafarevich theorem we obtain a universal bound for the cohomolgy with compact support of a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on a smooth curve over k with fixed rank and ramification bounded by a divisor of fixed degree. More precisely:

Definition 10.11. Let C be a smooth and geometrically connected curve over k . Let $d \geq 0$ be a natural number. Then we define

$$\mathcal{C}_d := 2g(\widehat{C}) + 2d + 1,$$

where $g(\widehat{C})$ is the genus of \widehat{C} . In case D is an effective divisor of degree $\deg(D) \geq 0$ on \widehat{C} we set

$$\mathcal{C}_D := \mathcal{C}_{\deg D}$$

and call it the *complexity* of D .

Proposition 10.12. Let C be a smooth geometrically connected curve over k . Let $r \geq 1$ and $d \geq 0$ be natural numbers. Then

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^0(C \otimes_k \bar{k}, \mathcal{F}) + \dim_{\bar{\mathbb{Q}}_\ell} H_c^1(C \otimes_k \bar{k}, \mathcal{F}) \leq r \cdot \mathcal{C}_d$$

for all lisse $\bar{\mathbb{Q}}_\ell$ -sheaves \mathcal{F} on $C \otimes_k \bar{k}$ of degree r whose ramification is bounded by $r \cdot D$, where D is an effective divisor of degree $\deg(D) = d$ and with support $\text{supp}(D) = \widehat{C} \otimes_k \bar{k} \setminus C \otimes_k \bar{k}$.

Proof. First assume $C = \widehat{C}$ is projective and hence $D = 0$. Let $x \in \widehat{C} \otimes_k \bar{k}$ be a closed point and denote by U the complement. Notice that $\text{Swan}_x(\mathcal{F}) = 0$. The exact sequence

$$\cdots \rightarrow H^i(\widehat{C} \otimes_k \bar{k}, \mathcal{F}) \rightarrow H^i(x, \mathcal{F}) \rightarrow H_c^{i+1}(U, \mathcal{F}) \rightarrow \cdots,$$

Theorem 9.1 and Lemma 9.8 give

$$\sum_i (-1)^i \dim_{\bar{\mathbb{Q}}_\ell} H^i(\widehat{C} \otimes_k \bar{k}, \mathcal{F}) = \chi_c(U, \mathcal{F}) + \dim_{\bar{\mathbb{Q}}_\ell} H^0(x, \mathcal{F}) \quad (10.5)$$

$$\begin{aligned} &= r\chi_c(U) + r \\ &= r\chi(\widehat{C} \otimes_k \bar{k}) \\ &= r(2 - 2g(\widehat{C})). \end{aligned} \quad (10.6)$$

Furthermore by Poicaré duality (Theorem 8.4)

$$\dim_{\bar{\mathbb{Q}}_\ell} H^2(\widehat{C} \otimes_k \bar{k}, \mathcal{F}) = \dim_{\bar{\mathbb{Q}}_\ell} H^0(\widehat{C} \otimes_k \bar{k}, \mathcal{F}) = \dim_{\bar{\mathbb{Q}}_\ell} \mathcal{F}_{\bar{\eta}}^{\pi_1(\widehat{C} \otimes_k \bar{k}, \bar{\eta})} \leq r,$$

where $\mathcal{F}_{\bar{\eta}}$ denotes the $\pi_1(\widehat{C} \otimes_k \bar{k}, \bar{\eta})$ -representation corresponding to \mathcal{F} . Thus (10.5) yields

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^0(C \otimes_k \bar{k}, \mathcal{F}) + \dim_{\bar{\mathbb{Q}}_\ell} H_c^1(C \otimes_k \bar{k}, \mathcal{F}) \leq r \cdot \mathcal{C}_0.$$

Now assume the inclusion $C \subseteq \widehat{C}$ is strict, i.e. C is affine and the ramification of \mathcal{F} is bounded by D with $\text{supp}(D) = \widehat{C} \otimes_k \bar{k} \setminus C \otimes_k \bar{k}$. Then by Corollary 8.5 and Poincaré duality 8.4

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^0(C \otimes_k \bar{k}, \mathcal{F}) = 0, \quad \dim_{\bar{\mathbb{Q}}_\ell} H_c^2(C \otimes_k \bar{k}, \mathcal{F}) = \dim_{\bar{\mathbb{Q}}_\ell} H^0(C \otimes_k \bar{k}, \mathcal{F}^\vee) \leq r.$$

Thus by Grothendieck-Ogg-Shafarevich 9.1 and Lemma 9.8

$$\begin{aligned} &\dim_{\bar{\mathbb{Q}}_\ell} H_c^1(C \otimes_k \bar{k}, \mathcal{F}) \\ &= r \cdot (2g(\widehat{C}) - 2) + r \cdot \text{card}(\text{supp}(D)) + \dim_{\bar{\mathbb{Q}}_\ell} H_c^2(C \otimes_k \bar{k}, \mathcal{F}) \\ &\quad + \sum_{x \in \text{supp}(D)} \text{Swan}_x(\mathcal{F}) \\ &\leq r \cdot (2g(\widehat{C}) - 2) + rd + r + rd \\ &= r(2g(\widehat{C}) + 2d - 1) \\ &< r \cdot \mathcal{C}_d. \end{aligned}$$

□

Definition 10.13. Let U be a normal k -scheme, A an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on U . We say that \mathcal{F} is *tame* if for any $\nu : C \rightarrow U$ in $\text{Cu}(U)$ we have $\text{Swan}(\nu^* \mathcal{F}) = 0$.

Remark 10.14. We immediately see that \mathcal{F} is tame if and only if there exists a divisorial compactification $U \hookrightarrow X$ with respect to which the ramification of \mathcal{F} is bounded by the zero divisor. Moreover, this is equivalent to the ramification of \mathcal{F} being bounded by the zero divisor with respect to all divisorial compactifications.

The following proposition is a direct consequence of Theorem 6.18 due to Kerz and Schmidt.

Proposition 10.15 (Kerz-Schmidt). *Let U be a normal connected scheme over k , $\bar{x} \rightarrow U$ a geometric point and A an ℓ -adic coefficient ring. Let $\pi_1^{\text{tame}}(U, \bar{x})$ be the tame fundamental group of U at \bar{x} , see Definition 6.20. Then the equivalence of categories from Theorem 7.13 restricts to an equivalence between the following full subcategories*

$$(\text{tame } A\text{-sheaves on } U) \xrightarrow{\cong} (A\text{-representations of } \pi_1^{\text{tame}}(U, \bar{x})).$$

Proof. In Theorem 7.13 we constructed two quasi-inverse functors

$$(\text{isse } A\text{-sheaves on } U) \xrightleftharpoons[G]{F} (A\text{-representations of } \pi_1(U, \bar{x}))$$

It suffices to show F and G induce functors between the full subcategories from the statement. In view of the construction of F and G it suffices to consider the case where A is a finite ring. Let \mathcal{F} be a tame A -sheaf on U . Then we have to show that the $\pi_1(U, \bar{x})$ -representation $F(\mathcal{F}) = \mathcal{F}_{\bar{x}}$ factors over the continuous quotient map $\pi_1(U, \bar{x}) \twoheadrightarrow \pi_1^{\text{tame}}(U, \bar{x})$, see Proposition 6.21, (a). By Proposition 7.6 the sheaf \mathcal{F} is represented by a finite étale U -group scheme $U_{\mathcal{F}}$. The stalk $\mathcal{F}_{\bar{x}}$ is equal to the fiber of $U_{\mathcal{F}} \rightarrow U$ at \bar{x} . By the definition of $\pi_1^{\text{tame}}(U, \bar{x})$ it suffices to prove that $U_{\mathcal{F}} \rightarrow U$ is tame, see Definition 6.15, (b). By Theorem 6.18 it suffices to check that for any smooth connected k -curve C and any map $\nu : C \rightarrow U$ the base changed morphism $U_{\mathcal{F}} \times_U C \rightarrow C$ is tamely ramified along $\widehat{C} \setminus C$. If $\nu(C)$ is a point there is nothing to show. Otherwise ν factors over the normalization $\widehat{\nu(C)} \rightarrow \nu(C) \subseteq U$. If the base change of $U_{\mathcal{F}} \rightarrow U$ over $\widehat{\nu(C)}$ is tame, then so is the base change over C , see e.g. [39, II, Cor. 7.8]. Thus we can assume that $\nu : C \rightarrow U$ is in $\text{Cu}(U)$ (see 10.0.1 for the Definition of $\text{Cu}(U)$). Notice that $U_{\nu^* \mathcal{F}} := U_{\mathcal{F}} \times_U C \rightarrow C$ is a finite étale morphism representing $\nu^* \mathcal{F}$. Since \mathcal{F} is tame we have $\text{Swan}_y(\nu^* \mathcal{F}) = 0$ for all $y \in \widehat{C}$, i.e. for $\bar{\eta} \rightarrow C$ a geometric point over the generic point of C and P_y the wild inertia group at y we have $0 = \mathcal{F}_{\bar{\eta}}^{P_y} = (U_{\mathcal{F}} \times_U \bar{\eta})^{P_y}$ (see 9.1.1, (c)). This means that $U_{\mathcal{F}} \times_U C \rightarrow C$ is tamely ramified.

It remains to show that if M is an A -representation of $\pi_1^{\text{tame}}(U, \bar{x})$ then the lisse sheaf $G(M)$ is a tame sheaf on U . But by construction of $\pi_1^{\text{tame}}(U, \bar{x})$ there is a finite étale Galois cover $P \rightarrow U$ which is tame such that $G(M)|_P$ is trivial. It follows that $G(M)$ is represented by a finite étale U -group-scheme $U_{G(M)}$ sitting inside $P \rightarrow U_{G(M)} \rightarrow U$. It follows that $U_{G(M)} \rightarrow U$ is tame. Hence for $\nu : C \rightarrow U$ in $\text{Cu}(U)$, the base change $U_{G(M)} \times_U C \rightarrow C$ is tame (by Theorem 6.18). This means $\text{Swan}(\nu^* G(M)) = 0$, so $G(M)$ is tame. \square

Remark 10.16. (a) Let A and U be as above and \mathcal{F} a lisse A -sheaf on U and \mathcal{L} a tame A -sheaf on U , which is free of rank 1. Let $U \hookrightarrow X$ be a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. Then the ramification of \mathcal{F} is bounded by D if and only if the ramification of $\mathcal{F} \otimes \mathcal{L}$ is bounded by D . Indeed if U is a curve and $x \in X \setminus U$ a point at infinity, then the restriction of the representation corresponding to \mathcal{L} to the wild inertia group at x is trivial; in particular $\text{Swan}_x(\mathcal{F}) = \text{Swan}_x(\mathcal{F} \otimes_A \mathcal{L})$. The general case follows from the curve case.

(b) If \mathcal{L} is a lisse A -sheaf on $\text{Spec } k$, then the pullback $\mathcal{L}|_U$ to U is tame.

10.1.1. *Weil sheaves.* (See [13, (1.1.7)-(1.1.12)].)

(a) Let \mathbb{F}_q be the finite field with $q = p^n$ elements and fix an algebraic closure \mathbb{F} . Denote by $F \in \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ the *geometric Frobenius* element given by $F(a) = a^{-q}$. We define the *Weil group* $W(\mathbb{F}/\mathbb{F}_q)$ of \mathbb{F}_q to be the subgroup of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ generated by F . Notice that under the

isomorphism $\text{Gal}(\mathbb{F}/\mathbb{F}_q) \xrightarrow{\sim} \hat{\mathbb{Z}}$ which sends F to 1, the group $W(\mathbb{F}/\mathbb{F}_q)$ is identified with \mathbb{Z} . It is equipped with the discrete topology.

- (b) Let X be a geometrically connected \mathbb{F}_q -scheme and $\bar{x} \rightarrow X \otimes_{\mathbb{F}_q} \mathbb{F}$ a geometric point. Then there is a short exact sequence (see [20, IX, Thm. 6.1])

$$0 \rightarrow \pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}(\mathbb{F}/\mathbb{F}_q) \rightarrow 0.$$

We define the *Weil group of X at \bar{x}* to be the topological group

$$W(X, \bar{x}) = \pi_1(X, \bar{x}) \times_{\text{Gal}(\mathbb{F}/\mathbb{F}_q)} W(\mathbb{F}/\mathbb{F}_q).$$

Notice that since $W(\mathbb{F}/\mathbb{F}_q)$ has the discrete topology the subgroup

$$\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{x}) = \pi_1(X, \bar{x}) \times_{\text{Gal}(\mathbb{F}/\mathbb{F}_q)} \{0\} \subseteq W(X, \bar{x})$$

is open and closed. (In particular, $W(X, \bar{x})$ is *not* equipped with the induced topology from its inclusion into $\pi_1(X, \bar{x})$, since $\pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{x}) \subseteq \pi_1(X, \bar{x})$ is not open, the quotient group being infinite.) We obtain a short exact sequence of topological groups

$$0 \rightarrow \pi_1(X \otimes_{\mathbb{F}_q} \mathbb{F}, \bar{x}) \rightarrow W(X, \bar{x}) \rightarrow W(\mathbb{F}/\mathbb{F}_q) \rightarrow 0.$$

- (c) Let X and $\bar{x} \rightarrow X \otimes_{\mathbb{F}_q} \mathbb{F}$ be as above. The action of $W(\mathbb{F}/\mathbb{F}_q)$ on \mathbb{F} induces an action of $W(\mathbb{F}/\mathbb{F}_q)$ on $X \otimes_{\mathbb{F}_q} \mathbb{F}$. By definition a *Weil sheaf on X* is a constructible $\bar{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} on $X \otimes_{\mathbb{F}_q} \mathbb{F}$ together with an action of $W(\mathbb{F}/\mathbb{F}_q)$ on \mathcal{F} , i.e. morphisms $\mathcal{F}(\sigma) : \mathcal{F} \rightarrow \sigma^* \mathcal{F}$, such that $\mathcal{F}(\text{id}) = \text{id}$ and $\tau^* \mathcal{F}(\sigma) \circ \mathcal{F}(\tau) = \mathcal{F}(\tau \circ \sigma)$. We say that \mathcal{F} is a *lisse Weil sheaf* if the underlying $\bar{\mathbb{Q}}_\ell$ -sheaf on $X \otimes_{\mathbb{F}_q} \mathbb{F}$ is lisse. Morphisms between Weil sheaves are morphisms between the underlying $\bar{\mathbb{Q}}_\ell$ -sheaves, which are compatible with the $W(\mathbb{F}/\mathbb{F}_q)$ -action in the obvious sense.

We record the following properties:

- (i) There are natural functors

$$\begin{aligned} (\text{constructible } \bar{\mathbb{Q}}_\ell\text{-sheaves on } X) &\hookrightarrow (\text{Weil sheaves on } X) \\ &\rightarrow (\text{constructible } \bar{\mathbb{Q}}_\ell\text{-sheaves on } X \otimes_{\mathbb{F}_q} \mathbb{F}). \end{aligned}$$

Here the first functor is given by pulling back a constructible sheaf on X to $X \otimes_{\mathbb{F}_q} \mathbb{F}$ with $W(\mathbb{F}/\mathbb{F}_q)$ acting via restriction of the natural $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ -action; this functor is fully faithful. The second functor is given by forgetting the $W(\mathbb{F}/\mathbb{F}_q)$ -action.

- (ii) Furthermore one can show that there is an equivalence of categories

$$(\text{lisse Weil sheaves on } X) \xrightarrow{\sim} (\bar{\mathbb{Q}}_\ell\text{-representations of } W(X, \bar{x})), \quad \mathcal{F} \mapsto \mathcal{F}_{\bar{x}}.$$

Here we define a $\bar{\mathbb{Q}}_\ell$ -representations of $W(X, \bar{x})$ as a group homomorphism $W(X, \bar{x}) \rightarrow \text{Aut}_{\bar{\mathbb{Q}}_\ell}(V)$ with V a finite dimensional $\bar{\mathbb{Q}}_\ell$ -vector space, which factors over a continuous homomorphism $W(X, \bar{x}) \rightarrow \text{Aut}_E(V_E)$, where E is a finite field extension of \mathbb{Q}_ℓ and V_E is a finite dimensional E -vector space with an isomorphism $V_E \otimes_E \bar{\mathbb{Q}}_\ell \cong V$.

- (d) Let U be a normal \mathbb{F}_q -scheme, $U \hookrightarrow X$ a divisorial compactification and D an effective Cartier divisor on X supported in $X \setminus U$. Let \mathcal{F} be a lisse Weil sheaf on U . Then we say that the *ramification of \mathcal{F} is bounded by D* if the ramification of the underlying lisse \mathbb{Q}_ℓ -sheaf on $U \otimes_{\mathbb{F}_q} \mathbb{F}$ is bounded by $D \otimes_{\mathbb{F}_q} \mathbb{F}$.

Remark 10.17. Notice that (up to isomorphism) a lisse Weil sheaf of rank 1 on $\text{Spec } \mathbb{F}_q$ is just a group homomorphism $W(\mathbb{F}/\mathbb{F}_q) = \mathbb{Z} \rightarrow \bar{\mathbb{Q}}_\ell^\times$, i.e. it corresponds uniquely to an element in $\bar{\mathbb{Q}}_\ell^\times$.

Theorem 10.18 (Deligne). *Let U be a smooth and connected \mathbb{F}_q -scheme, $U \hookrightarrow X$ a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. Let $r \geq 1$ be a natural number. Then, up to twist with a lisse rank 1 Weil sheaf coming from $\text{Spec } \mathbb{F}_q$, there are only finitely many irreducible lisse Weil sheaves of rank r on U whose ramification is bounded by D .*

In fact Deligne proves a more general finiteness result (see Theorem 10.31). We need some preparations to state it.

10.2. Skeleton sheaves.

10.2.1. *Semi-simplification.* Let G be a group (possibly infinite), E a field and V a finite dimensional E -vector space with G action, i.e. a representation of G . We can view V as a left $E[G]$ -module and $\dim_E V < \infty$ implies that it is artinian and noetherian. Therefore V has finite length as $E[G]$ -module, i.e. there is a chain of $E[G]$ -submodules

$$0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_r = V$$

such that the successive quotients V_i/V_{i-1} are simple (or irreducible) $E[G]$ -modules. Such a chain is called a decomposition series of V . By the Jordan-Hölder Theorem (see e.g. [8, I, §4.7, Thm. 6]) the length r is independent of the chosen decomposition series as is the sequence $V_r/V_{r-1}, \dots, V_1/V_0$, at least up to permutation and isomorphism. Therefore we can associate to V the $E[G]$ -module

$$V^{\text{ss}} = \bigoplus_{i=1}^r V_i/V_{i-1}.$$

The isomorphism class of this module depends only on V and it is called the *semi-simplification of V* . Notice that if G is a topological group which acts continuously on V , then the action of G on the subspaces V_i is automatically continuous (since the topology on the V_i is induced by the topology on V , cf. 7.1.11). The semi-simplification V^{ss} therefore is a continuous semi-simple G -representation.

In particular if \mathcal{F} is a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on a connected k -scheme X (resp. a lisse Weil sheaf), then we can view it as an $\bar{\mathbb{Q}}_\ell[\pi_1(X)]$ -module (resp. $\bar{\mathbb{Q}}_\ell[W(X)]$ -module) and can talk about its semi-simplification \mathcal{F}^{ss} . Notice that \mathcal{F}^{ss} is a direct sum of irreducible lisse $\bar{\mathbb{Q}}_\ell$ -sheaves (resp. lisse Weil sheaves).

Notation 10.19. Let U be a smooth and connected \mathbb{F}_q -scheme and $r \geq 1$ a natural number. We set

$$\mathcal{R}_r(U) = \{\text{isomorphism classes of lisse rank } r \text{ Weil sheaves on } U\} / \sim_{\text{ss}},$$

where \sim_{ss} is the equivalence relation defined by

$$\mathcal{F} \sim_{\text{ss}} \mathcal{G} : \Longleftrightarrow \mathcal{F}^{\text{ss}} \cong \mathcal{G}^{\text{ss}}.$$

Let $U \hookrightarrow X$ be a divisorial compactification and D an effective Cartier divisor supported on the complement. We set

$$\begin{aligned} \mathcal{R}_r(U, D) &:= \mathcal{R}_r(X, D) \\ &:= \{[\mathcal{F}] \in \mathcal{R}_r(U) \mid \text{the ramification of } \mathcal{F}^{\text{ss}} \text{ is bounded by } D\}, \end{aligned}$$

here $[\mathcal{F}]$ denotes the class of a lisse Weil sheaf \mathcal{F} of rank r in $\mathcal{R}_r(U)$.

Definition 10.20. Let U be a smooth connected \mathbb{F}_q -scheme.

- (a) A *skeleton sheaf* \mathcal{F} on U of rank r is a collection $\mathcal{F} = (\mathcal{F}_C)_{C \in \text{Cu}(U)}$, with $\mathcal{F}_C \in \mathcal{R}_r(C)$, which is compatible in the sense

$$\mathcal{F}_C|_{(C \times_U C')_{\text{red}}} = \mathcal{F}_{C'}|_{(C \times_U C')_{\text{red}}}, \quad \text{for all } C, C' \in \text{Cu}(U).$$

We denote by $\mathcal{V}_r(U)$ the set of rank r skeleton sheaves on U .

- (b) We say that a skeleton sheaf \mathcal{F} on U is *irreducible* if it cannot be written as a direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$, with $\mathcal{F}_i \in \mathcal{V}_{r_i}(U)$, $r_i \geq 1$. (For $C \in \text{Cu}(U)$ we have a well-defined notion of direct sum in $\bigcup_{r \geq 0} \mathcal{R}_r(C)$ and for $\mathcal{F}_i = (\mathcal{F}_{i,C})_{C \in \text{Cu}(U)} \in \mathcal{V}_{r_i}(U)$, $r_i \geq 1$, we define $\mathcal{F}_1 \oplus \mathcal{F}_2 = (\mathcal{F}_{1,C} \oplus \mathcal{F}_{2,C})_{C \in \text{Cu}(U)}$; clearly $\mathcal{F}_1 \oplus \mathcal{F}_2$ lies in $\mathcal{V}_{r_1+r_2}(U)$.)
- (c) Let $U \hookrightarrow X$ be a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. We say that a skeleton sheaf \mathcal{F} has *ramification bounded by D* if

$$\mathcal{F}_C \in \mathcal{R}_r(\widehat{C}, \nu_X^* D), \quad \text{for all } \nu : C \rightarrow U \text{ in } \text{Cu}(U).$$

We denote by $\mathcal{V}_r(U, D)$ or $\mathcal{V}_r(X, D)$ the set of rank r skeleton sheaves on U whose ramification is bounded by D .

Remark 10.21. Skeleton sheaves were introduced by Deligne and Drinfeld motivated by work of Wiesend, Kerz and Schmidt. See [16, 2.2] for a discussion where the name is coming from.

Proposition 10.22. Let U be a smooth connected \mathbb{F}_q -scheme and $r \geq 1$. There is a well defined map

$$\text{sk} : \mathcal{R}_r(U) \rightarrow \mathcal{V}_r(U), \quad [\mathcal{F}] \mapsto \text{sk}([\mathcal{F}]) := ([\mathcal{F}_C])_{C \in \text{Cu}(U)}.$$

It has the following properties:

- (a) If $U \hookrightarrow X$ is a divisorial compactification and D an effective Cartier divisor supported on $X \setminus U$, then sk restricts to a map $\mathcal{R}_r(X, D) \rightarrow \mathcal{V}_r(X, D)$.
- (b) The map sk is injective.
- (c) A lisse Weil sheaf \mathcal{F} on U is irreducible if and only if $\text{sk}([\mathcal{F}])$ is irreducible in the sense of Definition 10.20, (b).

Proof. For the well-definedness of sk we have to show that when we have two lisse Weil sheaves \mathcal{F}, \mathcal{G} of rank r on U with $\mathcal{F}^{\text{ss}} = \mathcal{G}^{\text{ss}}$, then $(\mathcal{F}_C)^{\text{ss}} = (\mathcal{G}_C)^{\text{ss}}$ for all $C \rightarrow U$ in $\text{Cu}(U)$. But if we identify \mathcal{F} with a representation of $\pi_1(U)$ and $0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_r = \mathcal{F}$ is a decomposition series of \mathcal{F} , then the chain

$0 = \mathcal{F}_{0|C} \subseteq \mathcal{F}_{1|C} \subseteq \dots \subseteq \mathcal{F}_{r|C} = \mathcal{F}_{|C}$ can be refined to a composition series of $\mathcal{F}_{|C}$. Hence

$$(\mathcal{F}_{|C})^{\text{ss}} \cong ((\mathcal{F}^{\text{ss}})_{|C})^{\text{ss}},$$

which immediately gives the well-definedness.

Part (a) of the Proposition follows directly from the definitions. The parts (b) and (c) require some preliminary interludes. We start with part (b), for which we recall some well-known facts:

10.2.2. Dirichlet density. We follow [44]. Let Y be an irreducible \mathbb{F}_q -scheme of dimension $n \geq 1$. Let $\|Y\|$ be the set of closed points in Y . Then the sum $\sum_{y \in \|Y\|} \frac{1}{q^{\deg(y) \cdot s}}$ converges absolutely for $s \in \mathbb{C}$ with $\text{Re}(s) > n$, where $\deg(y) = [\mathbb{F}_q(y) : \mathbb{F}_q]$. (This is a consequence of the fact that the zeta function $\zeta(Y, s)$ of Y converges in this domain.) Let $M \subseteq \|Y\|$ be a subset. We say that M has *Dirichlet density* δ if

$$\lim_{s \rightarrow n} \left(\frac{\sum_{y \in M} \frac{1}{q^{\deg(y) \cdot s}}}{\log\left(\frac{1}{s-n}\right)} \right) = \delta.$$

It follows from the fact that the zeta function of Y has a simple pole at $s = n$, that the Dirichlet density of $\|Y\|$ equals 1.

10.2.3. Let Y be an irreducible \mathbb{F}_q -scheme and $\bar{\eta} \rightarrow Y$ a geometric point over the generic point of Y . Let K be the residue field of the generic point of Y and $K \hookrightarrow \bar{K}$ the inclusion corresponding to $\bar{\eta} \rightarrow \eta$. Let \mathbb{F} be an algebraic closure of \mathbb{F}_q and denote by $F \in \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ the geometric Frobenius ($F(a) = a^{-q}$). For $y \in \|Y\|$ we denote by $F_y \in \pi_1(Y, \bar{\eta})$ the following conjugation class, called the *Frobenius class at y* : Denote by K_y the completion of K at y and choose an algebraic closure \bar{K}_y with an embedding $\bar{K} \hookrightarrow \bar{K}_y$. Denote by $I_y \subseteq \text{Gal}(\bar{K}_y/K_y)$ the inertia group. We have natural maps

$$\text{Gal}(\mathbb{F}/k(y)) \cong \text{Gal}(\bar{K}_y/K_y)/I_y \rightarrow \pi_1(Y, \bar{\eta})$$

and we define F_y to be the conjugacy class of the image of $F^{\deg(y)} \in \text{Gal}(\mathbb{F}/k(y))$ in $\pi_1(Y, \bar{\eta})$. If we choose a different embedding $\bar{K} \hookrightarrow \bar{K}_y$ then the image of $F^{\deg(y)}$ in $\pi_1(U, \bar{\eta})$ will differ by conjugation, so that the class F_y is independent of all the choices.

Notice that if we have a representation $\rho : \pi(Y, \bar{\eta}) \rightarrow \text{Aut}_A(M)$ where M is a finite free module over some ring A , then we can talk about the trace and the characteristic polynomial of $\rho(F_y)$.

Theorem 10.23 (Artin-Chebotarev density, see [44, Thm. 7]). *Let U be an irreducible \mathbb{F}_q -scheme, $\bar{\eta} \rightarrow U$ a geometric point over its generic point and $V \rightarrow U$ a pointed finite étale Galois covering with Galois group G . Let $S \subseteq G$ be a subset of G which is stable under conjugation. Then the set*

$$\{u \in \|U\| \mid F_u \text{ is mapped into } S \text{ under } \pi_1(U, \bar{\eta}) \twoheadrightarrow G\}$$

has Dirichlet density equal to $|S|/|G|$.

Corollary 10.24. *Let $\bar{\eta} \rightarrow U$ be as in Theorem 10.23 above. Then the subset $\bigcup_{u \in \|U\|} F_u$ is dense in $\pi_1(U, \bar{\eta})$.*

Proof. Since $\|U\|$ has Dirichlet density 1, Theorem 10.23 yields that $S := \bigcup_{u \in \|U\|} F_u$ maps surjectively to any finite quotient of $\pi_1(U, \bar{\eta})$. Since $\pi_1(U, \bar{\eta})$ is profinite the statement follows. \square

The following Proposition is a generalization of Corollary 4.26.

Proposition 10.25 ([9, §12, No. 1, Prop. 3]). *Let E be a field of characteristic zero and A an E -algebra with 1 (not necessarily commutative). Let M and M' be two semi-simple (left) A -modules which are finite dimensional as E -vector spaces. Then M and M' are isomorphic as A -modules if and only if $\text{Tr}_{M/E}(a) = \text{Tr}_{M'/E}(a)$, for all $a \in A$. Here $\text{Tr}_{M/E}(a)$ denotes the trace of the E -linear map $M \rightarrow M$, $m \mapsto a \cdot m$.*

Proof of Proposition 10.22, (b). First of all notice that for every $u \in \|U\|$ there is a curve $\nu : C \rightarrow U$ in $\text{Cu}(U)$ and a point $u_1 \in C$ such that ν induces an isomorphism $k(u) \xrightarrow{\sim} k(u_1)$. Indeed we can take a regular sequence $t_1, \dots, t_d \in \mathcal{O}_{U,u}$ which generates the maximal ideal. Then the vanishing locus of t_1, \dots, t_{d-1} defines a smooth curve C_0 in an open neighborhood of $u \in U$ and we can take $\nu : C \rightarrow U$ as the normalization of the closure of C_0 in U .

Now let \mathcal{F} and \mathcal{G} be two semi-simple lisse Weil sheaves on U of rank r and assume $\mathcal{F}|_C \cong \mathcal{G}|_C$, for all $\nu : C \rightarrow U$. By the above we get $\text{Tr}_{\mathcal{F}/\bar{\mathbb{Q}}_\ell}(F_u) = \text{Tr}_{\mathcal{G}/\bar{\mathbb{Q}}_\ell}(F_u)$ for all $u \in \|U\|$. (Here we identify \mathcal{F} and \mathcal{G} with their representations and use the notation from 10.2.3.) Therefore there exists a finite field extension E/\mathbb{Q}_ℓ such that $\text{Tr}_{\mathcal{F}/\bar{\mathbb{Q}}_\ell}, \text{Tr}_{\mathcal{G}/\bar{\mathbb{Q}}_\ell} : \pi_1(U) \rightarrow E$ are continuous class functions which by Corollary 10.24 coincide on a dense open subset. Since the ℓ -adic topology on E is Hausdorff this implies $\text{Tr}_{\mathcal{F}/\bar{\mathbb{Q}}_\ell} = \text{Tr}_{\mathcal{G}/\bar{\mathbb{Q}}_\ell}$. Now Proposition 10.25 with $A = E[\pi_1(U)]$ gives $\mathcal{F} = \mathcal{G}$. \square

Part (c) of Proposition 10.22 will be a consequence of a series of lemmas and propositions. We follow closely the arguments in [16, Prop. B.1]:

Lemma 10.26. *Let ℓ be a prime number and $K \rightarrow H$ a morphism of profinite groups with H a pro- ℓ group. Denote by H^{ab} the abelianization of H in the category of profinite groups; it is the quotient of H by the closure of the commutator subgroup. If the induced map $K \rightarrow H^{\text{ab}}/\ell H^{\text{ab}}$ is surjective, then so is $K \rightarrow H$.*

Proof. We may assume without loss of generality that we have an inverse system of maps between finite groups $K_i \rightarrow H_i$, $i \in I$, which in the limit gives the map $K \rightarrow H$. (Indeed we know $H = \varprojlim_U H/U$, where the limit is over all normal open subgroups $U \subseteq H$. We denote by K_U the preimage of U in K and can replace K by $\varprojlim_U K/K_U$.) Since $H^{\text{ab}} = \varprojlim_i H_i^{\text{ab}}$, we are reduced to the case in which K is a finite group and H a finite ℓ -group. Now assume the map $K \rightarrow H^{\text{ab}}/\ell$ is surjective. Since H^{ab} is a finitely generated $\mathbb{Z}/\ell^N \mathbb{Z}$ -module (for some $N \gg 0$), Nakayama's Lemma implies that $K \rightarrow H^{\text{ab}}$ is surjective.

Let ℓ^r be the order of H , then by [8, I, §6.5, Thm. 1] there exists a sequence

$$H = H^1 \supseteq H^2 \supseteq \dots \supseteq H^{r+1} = \{1\} \quad (10.7)$$

such that $[H, H^i] \subseteq H^{i+1}$, $1 \leq i \leq r$, and H^i/H^{i+1} is cyclic of order ℓ . Denote by $K' \subseteq H$ the image of K in H . Assume K' is strictly contained in H . Then there exists a minimal $i \leq r$ such that $N := K' \cdot H^{i+1} \subsetneq K' \cdot H^i = H$. Then $N \subseteq H$ is a normal subgroup. (Indeed it suffices to see that H^i normalizes N . It clearly normalizes H^{i+1} and for $k \in K'$ and $h \in H^i$ we have $h \cdot k \cdot h^{-1} = k \cdot [k^{-1}, h] \in N$.) Further H/N is a non-zero quotient of H^i/H^{i+1} hence is cyclic of order ℓ . Thus the quotient map $H \rightarrow H/N$ factors over $H^{\text{ab}} \rightarrow H/N$. But K maps to 1 in H/N , a contradiction to the surjectivity of $K \rightarrow H^{\text{ab}}$. Thus $K' = H$. \square

Lemma 10.27 ([16, Lem. B.2]). *Let U be a smooth connected \mathbb{F}_q -scheme and \mathcal{F} an irreducible lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on U . Then there exists a connected finite étale Galois cover $U' \rightarrow U$ with the following property: For any $\nu : C \rightarrow U$ in $\text{Cu}(U)$ such that $C \times_U U'$ is connected the pullback $\nu^* \mathcal{F}$ is irreducible.*

Proof. Take a DVR R finite over \mathbb{Z}_ℓ and a free lisse R -sheaf $(\mathcal{F}_n)_n$ such that $\mathcal{F} = (\mathcal{F}_n) \otimes_R \bar{\mathbb{Q}}_\ell$. Let $\mathbb{F}_\lambda = R/\mathfrak{m}$ be the residue field of R and denote by $\rho : \pi_1^{\text{ét}}(U) \rightarrow \text{GL}_r(R)$ the R -representation corresponding to (\mathcal{F}_n) , where $r = \text{rank } \mathcal{F}$. Let $G = \rho(\pi_1^{\text{ét}}(U))$ be the image of ρ and set

$$H_1 := \ker \left(\pi_1^{\text{ét}}(U) \xrightarrow{\rho} \text{GL}_r(R) \rightarrow \text{GL}_r(\mathbb{F}_\lambda) \right), \quad H_2 := \bigcap_{\varphi \in \text{Hom}_{\text{cont}}(H_1, \mathbb{Z}/\ell)} \ker(\varphi).$$

Then H_1 is an open normal subgroup of $\pi_1^{\text{ét}}(U)$, hence there exists a finite étale Galois cover $U_{H_1} \rightarrow U$ with $\pi_1^{\text{ét}}(U_{H_1}) = H_1$. Now the group H_1^{ab}/ℓ is profinite, in particular compact, and its Pontryagin dual is equal to

$$\text{Hom}_{\text{cont}}(H_1^{\text{ab}}/\ell, \mathbb{R}/\mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(U_{H_1}), \mathbb{Z}/\ell) = H^1(U_{H_1, \text{ét}}, \mathbb{Z}/\ell).$$

Here the second equality follows from the fact that for a finite abelian group A the first cohomology group $H^1(U_{\text{ét}}, A)$ is in bijection to the set of A -torsors over U (see e.g. [37, III, §4]). In particular H_1^{ab}/ℓ is a finite group, cf. 8.1.2, (a). Further, by definition $H_2 \subseteq H_1$ is a closed normal subgroup. We have $H_2 = \ker(H_1 \rightarrow H_1^{\text{ab}}/\ell)$. (Clearly \supseteq ; for the other inclusion it suffices to see that for any non-zero element $\sigma \in H_1^{\text{ab}}/\ell$ there exists a continuous map $H_1^{\text{ab}}/\ell \rightarrow \mathbb{Z}/\ell$ sending σ to a non-zero element. The construction of such a map is easily achieved by considering a filtration as in (10.7).) Hence $H_1/H_2 = H_1^{\text{ab}}/\ell$ and therefore H_2 is an open normal subgroup of H_1 . Denote by $U' \rightarrow U$ the associated Galois cover.

Now assume $\nu : C \rightarrow U$ is in $\text{Cu}(U)$ and $C' := C \times_U U'$ is connected. Then $C' \rightarrow C$ is a Galois cover with Galois group $\pi_1^{\text{ét}}(U)/H_2$. Denote by $H'_2 := \rho(H_2) \subseteq H'_1 := \rho(H_1)$ the images of H_i in G . Notice that H'_1 is a normal subgroup and is contained in $1 + \mathfrak{m} \text{End}_R(R^r)$, hence is pro- ℓ . Denote by ρ_C the composition (well-defined up to conjugation) $\rho_C : \pi_1^{\text{ét}}(C) \rightarrow \pi_1^{\text{ét}}(U) \rightarrow G$. By the above $\pi_1^{\text{ét}}(U)/H_2$ is a quotient of $\pi_1^{\text{ét}}(C)$. Hence ρ_C maps onto G/H'_2 and a fortiori onto G/H'_1 . Let $K \subseteq \pi_1^{\text{ét}}(C)$ be the preimage of $H'_1/H'_2 = H_1^{\text{ab}}/\ell$. Then $K \rightarrow H'_1$ is a map of profinite groups which is surjective by Lemma 10.26. All together we see that ρ_C is surjective. Since ρ is irreducible so is ρ_C and this finishes the proof. \square

We will need the following version of the Hilbert irreducibility theorem.

Theorem 10.28 (see e.g. [14, Cor. A.2]). *Let C be a smooth projective curve over \mathbb{F}_q with function field K . Let $x \in C$ be a closed point and denote by K_x the completion of K along x . Let $V \subseteq \mathbb{A}_K^n$ be a non-empty open subset and $V' \rightarrow V$ a connected finite étale covering. Then the set of K -rational points $y \in V(K)$ which do not split in V' (i.e. $y \times_V V'$ is a single point) is dense in $V(K_x)$. Here we equip $V(K_x) \subseteq K_x^n$ with the subspace topology.*

Lemma 10.29. *Let k be a perfect field and $y, y' \in \mathbb{A}_k^d$ two closed points. Then there exists an integral curve $\Gamma \subseteq \mathbb{A}_k^d$ with y, y' contained in the smooth locus of Γ .*

Proof. In case y, y' are k -rational points we can simply take Γ to be the line L connecting y and y' . In general we find a finite Galois extension k'/k such that y, y' split completely in $\mathbb{A}_{k'}^d$, i.e.

$$y \times_{\text{Spec } k} \text{Spec } k' = \{y_1, \dots, y_n\}, \quad y' \times_{\text{Spec } k} \text{Spec } k' = \{y'_1, \dots, y'_n\}$$

with k' -rational points $y_i, y'_j \in \mathbb{A}_{k'}^d$. Denote by $L_{i,j}$ the line in $\mathbb{A}_{k'}^d$ connecting y_i with y'_j . Set and let $\Gamma' = \cup_{i,j} L_{i,j}$; it is a curve in $\mathbb{A}_{k'}^d$, which is smooth in the points y_i, y'_j . Since $\text{Gal}(k'/k)$ acts via permutation on $\{y_i\}$ and on $\{y'_j\}$ it also permutes the $L_{i,j}$. In particular Γ' is stable under the $\text{Gal}(k'/k)$ action and by Galois descent (a special case of Theorem 7.7) there exists a curve $\Gamma_0 \subseteq \mathbb{A}_k^d$ with $\Gamma_0 \times_k k' = \Gamma'$. Let $\Gamma \subseteq \Gamma_0$ be an irreducible component (with its reduced scheme structure.) Then $\Gamma \times_k k'$ is a closed subcurve of Γ' , hence is the union of certain $L_{i,j}$ and therefore contains certain y_i, y'_j . It follows that Γ contains y, y' and that these lie in the smooth locus of Γ . \square

Proposition 10.30 ([16, Prop. B.1]). *Let U be a smooth connected \mathbb{F}_q -scheme, $x \in U$ a closed point and $\mathcal{F} \in \mathcal{R}_r(U)$ irreducible. Then there exists a curve $\nu : C \rightarrow U$ in $\text{Cu}(U)$ such that $x \in \nu(C)$ and $\nu^* \mathcal{F}$ is irreducible.*

Proof. By [13, Prop. (1.3.14)] there exists a character $\chi \in \mathcal{R}_1(\mathbb{F}_q)$ such that $\mathcal{F} \otimes \chi$ is a lisse $\bar{\mathbb{Q}}_\ell$ -sheaf. Thus we may assume from the beginning that \mathcal{F} is an irreducible lisse $\bar{\mathbb{Q}}_\ell$ -sheaf on U . Let $U' \rightarrow U$ be the finite étale Galois cover from Lemma 10.27. Then it suffices to construct a curve $\nu : C \rightarrow U$ in $\text{Cu}(U)$ such that $U' \times_U C$ is connected and $x \in \nu(C)$.

To this end we proceed as follows. By Noether normalization (see e.g. [15, Cor. 16.18]) there exists a finite and generically étale \mathbb{F}_q -morphism $f : U \rightarrow \mathbb{A}_{\mathbb{F}_q}^d$. Set $y := f(x) \in \mathbb{A}_{\mathbb{F}_q}^d$. Denote by $V \subseteq \mathbb{A}_{\mathbb{F}_q}^d$ a non-empty open subset such that $f^{-1}(V) \rightarrow V$ is finite étale. Let $y' \in V$ be a closed point. Then by Lemma 10.29 we find $\mu : \Gamma \rightarrow \mathbb{A}_{\mathbb{F}_q}^d$ in $\text{Cu}(\mathbb{A}_{\mathbb{F}_q}^d)$, such that μ is an isomorphism over y, y' . Let K be the function field of Γ and K_y the completion along y . Since Γ is smooth the choice of a local parameter at y yields a unique isomorphism of \mathbb{F}_q -algebras $K_y \cong k(y)((t))$ (see e.g. [46, II, §4]). On the other hand $k(y)$ is a finite extension of \mathbb{F}_q hence of the form $k(y) = \mathbb{F}_q[x]/f$ for some irreducible polynomial f . Thus there is a closed point $z \in \mathbb{A}_k^1$ with $k(z) = k(y)$ and the completion of $k(\mathbb{A}^1)$ at z is isomorphic to K_y , i.e. $k(\mathbb{A}^1)_z \cong K_y = k(z)((t))$.

Let $\mathbb{A}_{\mathbb{F}_q}^d \rightarrow \mathbb{A}_{\mathbb{F}_q}^1$ be a linear projection such that the composition $\Gamma \rightarrow \mathbb{A}_{\mathbb{F}_q}^d \rightarrow \mathbb{A}_{\mathbb{F}_q}^1$ is finite. Denote by $\mathbb{A}_{k(\mathbb{A}^1)}^{d-1}$ the base change of this projection

along $\mathrm{Spec} k(\mathbb{A}^1) \rightarrow \mathbb{A}_{\mathbb{F}_q}^1$ (inclusion of the generic point) and by $V_{k(\mathbb{A}^1)} \subseteq \mathbb{A}_{k(\mathbb{A}^1)}^{d-1}$ the corresponding base change of V . Since Γ is finite over \mathbb{A}^1 and $\mu(\Gamma) \cap V \ni y'$ the map μ induces a morphism $\mathrm{Spec} k(\mathbb{A}^1)_z = \mathrm{Spec} K_y \rightarrow V_{k(\mathbb{A}^1)}$, i.e. an element $\mu_z \in V_{k(\mathbb{A}^1)}(k(\mathbb{A}^1)_z)$. By Theorem 10.28 we find a point $v \in V_{k(\mathbb{A}^1)}(k(\mathbb{A}^1))$ which is t -adically arbitrarily close to μ_z and which does not split in $U'_{k(\mathbb{A}^1)}$. Since μ_z spreads out to $\mathrm{Spec}(k(z)[[t]]) = \mathrm{Spec} \mathcal{O}_{K_y} \rightarrow \mathrm{Spec} \mathcal{O}_{\mathbb{A}_{\mathbb{F}_q}^d, y}$, where the closed point maps to y , we can achieve that v spreads out to a map $\mathrm{Spec} \mathcal{O}_{\mathbb{A}^1, z} \rightarrow \mathrm{Spec} \mathcal{O}_{\mathbb{A}_{\mathbb{F}_q}^d, y}$, which sends the closed point to y . Then v defines a point in V whose closure in $\mathbb{A}_{\mathbb{F}_q}^d$ is a curve containing y and over which in U' – a fortiori in U – lies exactly one point. Let $u \in U$ be the unique point lying over v . By the going-down theorem (see e.g. [36, Thm. 9.4, (ii)]) there exists a curve in U whose generic point lies over v and which contains x ; since u is the only point in U mapping to v we get that the closure of u in U contains x . Let $\nu : C \rightarrow U$ be the normalization of $\overline{\{u\}}$. Then $x \in \nu(C)$ and $U' \times_U C$ is irreducible by the choice of v . This finishes the proof. \square

Proof of Proposition 10.22, (c). Let \mathcal{F} be a lisse Weil sheaf of rank r on U . If \mathcal{F} is not irreducible, then the class $[\mathcal{F}]$ in $\mathcal{R}_r(U)$ can be written as a non-trivial sum $[\mathcal{F}] = [\mathcal{F}_1] \oplus [\mathcal{F}_2]$. Hence $\mathrm{sk}([\mathcal{F}]) = \mathrm{sk}([\mathcal{F}_1]) \oplus \mathrm{sk}([\mathcal{F}_2])$ is not irreducible in $\mathcal{V}_r(U)$. If \mathcal{F} is irreducible, then for all closed points $x \in U$ there exists a curve $\nu_x : C_x \rightarrow U$ such that $x \in \nu_x(C_x)$ and $\nu_x^* \mathcal{F}$ is irreducible, by Proposition 10.30. Assume $\mathrm{sk}([\mathcal{F}]) = \mathcal{F}_1 \oplus \mathcal{F}_2$, with $\mathcal{F}_i \in \mathcal{V}_{r_i}(U)$. Then it follows that either $\nu_x^* \mathcal{F}_1 = 0$ for all $x \in U$ or $\nu_x^* \mathcal{F}_2 = 0$ for all $x \in U$. (Indeed the set Z_i of all $x \in U$ such that $\nu_x^* \mathcal{F}_i = 0$ is closed; since $U = Z_1 \sqcup Z_2$ and U is connected we have either $Z_1 = U$ or $Z_2 = U$.) As in the proof of Proposition 10.22, (b), Artin-Chebotarev density (Corollary 10.24) yields that either $\mathcal{F}_1 = 0$ or $\mathcal{F}_2 = 0$. Hence $\mathrm{sk}([\mathcal{F}])$ is irreducible. This finishes the proof of Proposition 10.22. \square

In view of Proposition 10.22, Theorem 10.18 is a consequence of the following theorem.

Theorem 10.31 (Deligne). *Let U be a smooth connected \mathbb{F}_q -scheme, $U \hookrightarrow X$ a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. Then the set of irreducible skeleton sheaves $\mathcal{F} \in \mathcal{V}_r(X, D)$ is finite up to twists by elements from $\mathcal{R}_1(\mathrm{Spec} \mathbb{F}_q)$ and its cardinality does not depend on the choices of $\ell \neq p$.*

For a proof of this Theorem, which relies on the Langlands correspondence [29] and Weil II [13], we refer the reader to [16].

Question 10.32 (Deligne, see [16, Question 1.2]). Let U be a smooth connected \mathbb{F}_q -scheme, $U \hookrightarrow X$ a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. Is the injection $\mathrm{sk} : \mathcal{R}_r(X, D) \hookrightarrow \mathcal{V}_r(X, D)$ actually a bijection?

Remark 10.33. The above question has a positive answer for $r = 1$, see [26, Cor. V].

11. GROTHENDIECK-OGG-SHAFAREVICH IN HIGHER DIMENSIONS FOLLOWING KATO-SAITO

In this section we give a short account of the generalization of the Grothendieck-Ogg-Shafarevich formula to higher dimension due to Kato and Saito.

11.0.1. Throughout this section we fix the following notation:

- k is a perfect field of characteristic $p > 0$ and \bar{k} an algebraic closure.
- ℓ is a prime number different from p .
- A k -scheme is a scheme which is separated and of finite type over k .
- If X is a k -scheme we denote by $CH_0(X)$ the Chow group of zero-cycles on X modulo rational equivalence, i.e. it is the free abelian group on the set of closed points of X modulo the subgroup generated by elements of the form $\text{div}(f)_C$, where $C \subseteq X$ is a closed integral subscheme of dimension 1, $f \in k(C)^\times$ and $\text{div}(f)_C$ denotes the divisor on C associated to f , which we view as a zero-cycle on X .

11.1. Higher dimensional version of Grothendieck-Ogg-Shafarevich.

11.1.1. *Intersection product with the log-diagonal.* Let U be a smooth connected k -scheme of dimension d . Let $j : U \hookrightarrow X$ and $j' : U \hookrightarrow X'$ be two open embeddings of U into integral proper k -schemes. If there exists a morphism $\pi : X' \rightarrow X$ such that $\pi \circ j' = j$, then π is unique and induces a proper morphism $\pi : X' \setminus U \rightarrow X \setminus U$ and hence also a pushforward

$$\pi_* : CH_0(X' \setminus U) \rightarrow CH_0(X \setminus U),$$

which sends the class of a closed point $x' \in X' \setminus U$ to the class of $[x' : \pi(x')] \cdot \pi(x')$. We obtain a projective system indexed by open embeddings $j : U \hookrightarrow X$ as above. We set

$$CH_0(\partial U) := \varprojlim_j CH_0(X \setminus U), \quad CH_0(\partial U)_{\mathbb{Q}} := \varprojlim_j (CH_0(X \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

The degree map $\deg_k : CH_0(X \setminus U) \rightarrow CH_0(\text{Spec } k) = \mathbb{Z}$ is induced by pushforward along the structure map $X \rightarrow \text{Spec } k$ and hence induces well-defined degree maps

$$\deg_k : CH_0(\partial U) \rightarrow \mathbb{Z}, \quad \deg_k : CH_0(\partial U)_{\mathbb{Q}} \rightarrow \mathbb{Q}.$$

Notice that if U is a curve with unique smooth compactification $j : U \hookrightarrow C$, then $CH_0(\partial U) = CH_0(C \setminus U)$ and $CH_0(\partial U)_{\mathbb{Q}} = CH_0(C \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $V \rightarrow U$ be a finite étale Galois cover with Galois group G . Denote by $\Delta_V \subseteq V \times_U V$ the diagonal. Notice that

$$(V \times_U V) \setminus \Delta_V = \bigsqcup_{\sigma \in G \setminus \{1\}} \Gamma_{\sigma},$$

where Γ_{σ} is the graph of $\sigma : V \rightarrow V$. In [22, Thm 3.2.3] Kato-Saito define for each $\sigma \neq 1$ an element

$$(\Gamma_{\sigma}, \Delta_V)^{\log} \in CH_0(\partial V)_{\mathbb{Q}},$$

which is called *the intersection product of Γ_{σ} with the log-diagonal*. It has the following properties:

- (a) The subscheme $\Gamma_\sigma \subseteq V \times_U V$ defines a map

$$\Gamma_\sigma^* : H_c^i(V \otimes_k \bar{k}, \mathbb{Q}_\ell) \xrightarrow{p_1^*} H_c^i(\Gamma_\sigma \otimes_k \bar{k}, \mathbb{Q}_\ell) \xrightarrow{p_{2*}} H_c^i(V \otimes_k \bar{k}, \mathbb{Q}_\ell).$$

Here the $p_i : \Gamma_\sigma \rightarrow V$, $i = 1, 2$, are induced by the two projection maps $V \times_U V \rightarrow V$ and the pushforward p_{2*} is induced by

$$p_{2*}(\mathbb{Q}_{\ell, \Gamma_\sigma}) = (\mathbb{Q}_{\ell, V})^{|G|} \xrightarrow{\Sigma} \mathbb{Q}_{\ell, V}.$$

Then (see [22, Prop. 3.2.4])

$$\sum_{i=0}^{2d} (-1)^i \text{Tr}(\Gamma_\sigma^* | H_c^i(V \otimes_k \bar{k}, \mathbb{Q}_\ell)) = \deg_k(\Gamma_\sigma, \Delta_V)^{\log}.$$

- (b) Assume there exist compactifications $V \hookrightarrow Y$ and $U \hookrightarrow X$ and an étale morphism $Y \rightarrow X$ extending the covering $V \rightarrow U$. Then the image of $(\Gamma_\sigma, \Delta_V)^{\log}$ under the projection

$$CH_0(\partial V)_{\mathbb{Q}} \rightarrow CH_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q} \quad (11.1)$$

is zero (see [22, Cor. 3.3.4]).

- (c) Let $V \hookrightarrow Y$ and $U \hookrightarrow X$ be compactifications and $Y \rightarrow X$ a morphism extending $V \rightarrow U$. Assume X is smooth and $X \setminus U$ is a strict normal crossings divisor. If $Y \rightarrow X$ is tamely ramified along $X \setminus U$ (in the sense of Definition 6.15, (a)), then $(\Gamma_\sigma, \Delta_V)^{\log}$ maps to zero under the projection (11.1) (see [22, Prop. 3.3.5, 2.]).

11.1.2. *Swan character class and Swan class.* Let U be a smooth connected k -scheme.

- (a) Let $\pi : V \rightarrow U$ be a finite étale Galois covering with Galois group G . Then in [22, Def. 4.1.1] the *Swan character* $s_{V/U}(\sigma) \in CH_0(\partial V)$, $\sigma \in G$, is defined by

$$s_{V/U}(\sigma) = \begin{cases} -(\Gamma_\sigma, \Delta_V)^{\log}, & \text{if } \sigma \neq 1 \\ \sum_{\sigma \neq 1} (\Gamma_\sigma, \Delta_V)^{\log}, & \text{if } \sigma = 1. \end{cases}$$

- (b) Let \mathcal{F} be a lisse \mathbb{F}_λ -sheaf on U , for some finite extension \mathbb{F}_λ of \mathbb{F}_ℓ . Assume \mathcal{F} is trivialized by a Galois covering $\pi : V \rightarrow U$ as in (a). Then in [22, Def. 4.2.1] the *Swan class* $\text{Sw}_{V/U}(\mathcal{F}) \in CH_0(\partial V)_{\mathbb{Q}}$ is defined by

$$\text{Sw}_{V/U}(\mathcal{F}) := \sum_{\sigma \in G_{(p)}} \left(\dim_{\mathbb{F}_\ell} M^\sigma - \frac{\dim_{\mathbb{F}_\ell} M^{\sigma^p} / M^\sigma}{p-1} \right) \cdot s_{V/U}(\sigma).$$

Here M is the G -representation corresponding to \mathcal{F} , $M^\sigma = \{m \in M \mid \sigma(m) = m\}$ and $G_{(p)} = \{\sigma \in G \mid \sigma \text{ has } p\text{-power order}\}$.

- (c) Let A be an ℓ -adic coefficient ring and \mathcal{F} a lisse A -sheaf on U . Then in [22, Def. 4.2.8] the *Swan class* $\text{Sw}_U(\mathcal{F}) \in CH_0(\partial U)_{\mathbb{Q}}$ is defined as follows: We find a DVR R finite over \mathbb{Z}_ℓ and a lisse R -sheaf (\mathcal{F}_n) such that $\mathcal{F} = (\mathcal{F}_n) \otimes_R A$. We find a Galois cover $\pi : V \rightarrow U$ as in (a) trivializing \mathcal{F}_1 and we define

$$\text{Sw}_U(\mathcal{F}) := \frac{1}{|G|} \cdot \pi_* \text{Sw}_{V/U}(\mathcal{F}_1).$$

This definition is independent of the choice of R , (\mathcal{F}_n) , and π .

- Remark 11.1.** (a) Considering Lemma 9.6, the definition of $s_{V/U}(\sigma)$ above should remind you of the definition of the character a_G in Theorem 4.44.
- (b) If U is a smooth curve with smooth compactification X and \mathcal{F} is a lisse A -sheaf on U , then $\text{Sw}_U(\mathcal{F}) = \sum_{x \in X \setminus U} \text{Swan}_x(\mathcal{F})[x]$ in $CH_0(\partial U)_{\mathbb{Q}} = CH_0(X \setminus U) \otimes_{\mathbb{Z}} \mathbb{Q}$, see [22, Introduction].
- (c) In [22, Conj. 4.3.7] it is conjectured that $\text{Sw}_U(\mathcal{F})$ always lies in the image of $CH_0(\partial U) \rightarrow CH_0(\partial U)_{\mathbb{Q}}$. In dimension 1 this is a consequence of the Hasse-Arf theorem and the remark above. In dimension 2 this conjecture is proved in [22, Cor. 5.1.7].

The higher dimensional version of the Grothendieck-Ogg-Shafarevich formula now takes the following form:

Theorem 11.2 ([22, 4.2.9]). *Let U be a smooth and connected k -scheme of dimension d and \mathcal{F} a lisse $\bar{\mathbb{Q}}_{\ell}$ -sheaf on U . Then*

$$\begin{aligned} \chi_c(U \otimes_k \bar{k}, \mathcal{F}) &:= \sum_{i=0}^{2d} (-1)^i \dim_{\bar{\mathbb{Q}}_{\ell}} H_c^i(U \otimes_k \bar{k}, \mathcal{F}) \\ &= \text{rk}(\mathcal{F}) \cdot \chi_c(U, \bar{\mathbb{Q}}_{\ell}) - \deg_k(\text{Sw}_U(\mathcal{F})). \end{aligned}$$

We conclude with some remarks on more recent developments.

11.2. Outlook on ramification theory following Abbes-Saito.

- (a) In [3], Abbes and Saito give a refinement of Theorem 11.2. If $\pi : X \rightarrow \text{Spec } k$ is a k -scheme and A an ℓ -adic coefficient ring, then the dualizing complex $K_X := \pi^!(A_{\text{Spec } k})$ is defined as an object in the derived category of complexes of A -modules of finite tor-dimension with constructible cohomology. (In case π is smooth of pure dimension d , $K_X \cong A(d)[2d]$.) Let \mathcal{F} be a constructible sheaf of free A -modules. Then in [3, Def. 2.1.1] the *characteristic class* of \mathcal{F} is defined as an element $C(\mathcal{F}) \in H^0(X, K_X)$. It has the property that if π is proper, then

$$\text{Tr}_{\pi}(C(\mathcal{F})) = \chi(X_{\bar{k}}, \mathcal{F}) \in A$$

where $\text{Tr}_{\pi} : H^0(X, K_X) \rightarrow H^0(\text{Spec } k, A) = A$ is the trace map.

Now assume U is a smooth k -scheme and that $j : U \hookrightarrow X$ is a compactification. Let \mathcal{F} be a free lisse A -sheaf on U which is of Kummer type with respect to X (see [3, Def. 3.1.1]). Then ([3, Thm. 3.3.1])

$$C(j_! \mathcal{F}) = \text{rk}(\mathcal{F}) \cdot C(j_! A_U) - [\text{Sw}_U^{\text{naive}}(\mathcal{F})] \quad \text{in } H^0(X, K_X), \quad (11.2)$$

where $\text{Sw}_U^{\text{naive}}(\mathcal{F})$ is the naive Swan class (it is conjectured to be equal to $\text{Sw}_U(\mathcal{F})$ and it is known that they have the same degree) and $[\text{Sw}_U^{\text{naive}}(\mathcal{F})]$ denotes the cycle class of $\text{Sw}_U^{\text{naive}}(\mathcal{F})$ in $H^0(X, K_X)$. In particular applying Tr_{π} we get back Theorem 11.2 for \mathcal{F} as above. Actually, Theorem 11.2 can be reproved in its full generality using the equality (11.2), see [3, Cor. 3.3.2].

- (b) We comment on the relation between the ramification theory via curves which is described in Section 10 and the ramification theory developed by Abbes-Saito in [1], [2], [42], [4].

Let K be a complete discrete valuation field with arbitrary residue field, K^{sep} a separable closure of K and $G = \text{Gal}(K^{\text{sep}}/K)$ the absolute Galois group. Then in [1] a *logarithmic ramification filtration* $(G_{\log}^r)_{r \in \mathbb{Q}_{\geq 0}}$ is defined, which has the following properties:

- (a) $G_{\log}^s \subseteq G_{\log}^r$, for $s < r$.
- (b) $G_{\log}^0 = I = \text{inertia subgroup of } G$, $G_{\log}^{0+} := \overline{\bigcup_{r>0} G_{\log}^r} = P = \text{wild inertia subgroup}$.
- (c) Set $G_{\log}^{r+} := \overline{\bigcup_{s>r} G_{\log}^s}$. If K has characteristic $p > 0$ and “comes from geometry”, then the quotient G_{\log}^r/G_{\log}^{r+} is abelian and is killed by p for all $r > 0$.
- (d) If the residue field of K is perfect, this filtration coincides with the classical ramification filtration from Definition 3.54.

Now let U be a smooth connected k -scheme and denote by $\bar{\eta} \rightarrow U$ a geometric point over the generic point of U . Assume that we have a dominant open embedding $j : U \hookrightarrow X$ into a smooth k -scheme such that the complement $X \setminus U$ with its reduced structure is a strict normal crossings divisor $D = \cup_i D_i$. Denote the generic points of D by $\xi_i \in D_i$ and by G_{ξ_i} the decomposition subgroups at ξ_i of the absolute Galois group of $k(X)$ (well defined up to conjugation). Let \mathcal{F} be a free lisse A -sheaf on U . Abbes and Saito define in [4, Def. 8.10, (ii)] the *(log) conductor of \mathcal{F} relative to X* to be the rational effective divisor supported in D

$$\text{Cond}^{\log}(\mathcal{F}) := \sum_i \text{cond}_{\xi_i}^{\log}(\mathcal{F}) \cdot D_i,$$

where $\text{cond}_{\xi_i}^{\log}(\mathcal{F}) \in \mathbb{Q}_{\geq 0}$ is the minimum over all $r \geq 0$ such that $G_{\xi_i, \log}^{r+}$ acts trivial on the $\pi_1(U, \bar{\eta})$ -representation $\mathcal{F}_{\bar{\eta}}$.

In [16, after Remark 3.10] the following conjecture is formulated (see also [6, Conj. B]):

Conjecture 11.3. *Let U be a smooth connected k -scheme and \mathcal{F} a lisse A -sheaf on U . Let $j : U \hookrightarrow X$ be a divisorial compactification and D an effective Cartier divisor supported in $X \setminus U$. Then the following two statements are equivalent:*

- (a) *The ramification of \mathcal{F} is bounded by D in the sense of Definition 10.4.*
- (b) *For all k -morphisms $h : V \rightarrow X$ fitting into a commutative diagram*

$$\begin{array}{ccc} & & V \\ & \nearrow^{j'} & \downarrow h \\ U & \xrightarrow{j} & X, \end{array}$$

such that V is a smooth k -scheme, j' is an open dominant embedding and the complement $V \setminus U$ is a strict normal crossings divisor, we have

$$\text{Cond}^{\log}(h^* \mathcal{F}) \leq h^* D.$$

For $D = 0$ this is Proposition 10.15, which is a direct consequence of Theorem 6.18 by Kerz and Schmidt; for X smooth and \mathcal{F} of rank 1 this conjecture is proved in [6, Thm. 7.1].

REFERENCES

- [1] A. Abbes and T. Saito, *Ramification of local fields with imperfect residue fields*, Amer. J. Math. **124** (2002), no. 5, 879–920.
- [2] ———, *Ramification of local fields with imperfect residue fields. II*, Doc. Math. (2003), no. Extra Vol., 5–72 (electronic), Kazuya Kato’s fiftieth birthday.
- [3] ———, *The characteristic class and ramification of an l -adic étale sheaf*, Invent. Math. **168** (2007), no. 3, 567–612.
- [4] ———, *Ramification and cleanliness*, Tohoku Math. J. (2) **63** (2011), no. 4, 775–853.
- [5] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas: Séminaire de Géométrie Algébrique de Bois-Marie 1963/1964.*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer-Verlag, 1972/1973.
- [6] I. Barrientos, *Log ramification via curves in rank 1*, Preprint 2013, <http://arxiv.org/abs/1307.5814>.
- [7] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 9*, Springer, Berlin, 2007, reprint of the 1959 original.
- [8] ———, *Éléments de mathématique. Algèbre. Chapitres 1 à 3*, Springer, Berlin, 2007, reprint of the 1970 original.
- [9] ———, *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*, Springer, Berlin, 2012, second revised edition of the 1958 edition.
- [10] J.W.S. Cassels and A. Fröhlich (eds.), *Algebraic number theory. Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union (held at the University of Sussex, Brighton, September 1-17, 1965). 1st paperback ed., 2nd printing.*, 1987 (English).
- [11] B. Conrad, *Deligne’s notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257.
- [12] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4^{1/2}, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.
- [13] ———, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252.
- [14] V. Drinfeld, *On a conjecture of Deligne*, Mosc. Math. J. **12** (2012), no. 3, 515–542, 668.
- [15] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.
- [16] H. Esnault and M. Kerz, *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Math. Vietnam. **37** (2012), no. 4, 531–562.
- [17] P. Gille, *Le groupe fondamental sauvage d’une courbe affine en caractéristique $p > 0$* , Courbes semi-stables et groupe fondamental en géométrie algébrique (Luminy, 1998), Progr. Math., vol. 187, Birkhäuser, Basel, 2000, pp. 217–231.
- [18] A. Grothendieck, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas.*, Publ. Math. IHES **32** (1967).
- [19] A. Grothendieck, L. Illusie, J.-P. Juanolou, and I. Bucur, *Cohomologie ℓ -adique et fonctions L : Séminaire de Géométrie Algébrique de Bois-Marie 1965/1966*, Lecture Notes in Mathematics, Springer-Verlag, 1977.
- [20] A. Grothendieck and M. Raynaud, *Revêtements Étales et Groupe Fondamental: Séminaire de Géométrie Algébrique de Bois-Marie 1960/61.*, Lecture Notes in Mathematics, vol. 224, Springer-Verlag, 1971.
- [21] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

- [22] K. Kato and T. Saito, *Ramification theory for varieties over a perfect field*, Ann. of Math. (2) **168** (2008), no. 1, 33–96.
- [23] N. M. Katz, *Wild ramification and some problems of “independence of l ”*, Amer. J. Math. **105** (1983), no. 1, 201–227.
- [24] ———, *Local-to-global extensions of representations of fundamental groups*, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 4, 69–106.
- [25] ———, *Gauss sums, Kloosterman sums, and monodromy groups*, Annals of Mathematics Studies, vol. 116, Princeton University Press, Princeton, NJ, 1988.
- [26] M. Kerz and S. Saito, *Chow group of 0-cycles with modulus and higher dimensional class field theory*, Preprint 2013, <http://arxiv.org/abs/1304.4400>.
- [27] M. Kerz and A. Schmidt, *On different notions of tameness in arithmetic geometry*, Math. Ann. **346** (2010), no. 3, 641–668.
- [28] J. Kollár, *Fundamental groups of rationally connected varieties*, Michigan Math. J. **48** (2000), 359–368, Dedicated to William Fulton on the occasion of his 60th birthday.
- [29] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002), no. 1, 1–241.
- [30] T. Y. Lam, *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.
- [31] G. Laumon, *Semi-continuité du conducteur de Swan (d’après P. Deligne)*, The Euler-Poincaré characteristic (French), Astérisque, vol. 83, Soc. Math. France, Paris, 1981, pp. 173–219.
- [32] ———, *Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. (1987), no. 65, 131–210.
- [33] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Ern , Oxford Science Publications.
- [34] W. L tkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), no. 1, 95–111.
- [35] H. Matsumura, *Commutative algebra*, W. A. Benjamin, Inc., New York, 1970.
- [36] ———, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.
- [37] J. S. Milne, * tale cohomology*, Princeton Mathematics Series, no. 33, Princeton University Press, Princeton, 1980.
- [38] James S. Milne, *Fields and galois theory (v4.50)*, 2014, Available at www.jmilne.org/math/, p. 138.
- [39] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften, vol. 322, Springer-Verlag, Berlin, 1999.
- [40] N. Nikolov and D. Segal, *On finitely generated profinite groups. I. Strong completeness and uniform bounds*, Ann. of Math. (2) **165** (2007), no. 1, 171–238.
- [41] M. Raynaud, *Caract ristique d’Euler-Poincar  d’un faisceau et cohomologie des vari t s ab liennes*, S minaire Bourbaki, Vol. 9, Soc. Math. France, Paris, 1995, pp. Exp. No. 286, 129–147.
- [42] T. Saito, *Wild ramification and the characteristic cycle of an l -adic sheaf*, J. Inst. Math. Jussieu **8** (2009), no. 4, 769–829.
- [43] J.-P. Serre, *Sur la rationalit  des repr sentations d’Artin*, Ann. of Math. (2) **72** (1960), 405–420.
- [44] ———, *Zeta and L functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1965, pp. 82–92.
- [45] ———, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.
- [46] ———, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg.
- [47] ———, *Galois cohomology*, english ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author.

- [48] T. Szamuely, *Galois groups and fundamental groups*, Cambridge Studies in Advanced Mathematics, vol. 117, Cambridge University Press, Cambridge, 2009.
- [49] P. Webb, *Finite group representations for the pure mathematician*, 2012, course notes available at <http://www.math.umn.edu/~webb/>.

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